Calculating theoretical constraints of positivity bounds on effective field theories for particle physics, the Proca EFT and SMEFTs

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Abstract

This project investigates the various applications of theoretical positivity bounds due to the unitarity and analycity conditions on particle physics, significantly limiting the parameter space allowed for IR theories in the forward scattering limit. Assuming local, unitary and causal operators results in positivity for certain EFT parameters such as coupling constants, and explicitly calculating these constraints is the main objective of this project. The considered theories include a U(1) scalar theory, the Proca EFT vector field and a quartic gauge boson scattering process in the Standard Model, offering applications to numerous related fields within theoretical and experimental physics.

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Acknowledgements

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Chapter 1

Introduction

It has always been the aim of particle physics to describe the matter making up our universe in a way that makes its underlying, fundamental symmetries apparent and show the beauty of these symmetries, not only to the trained eyes of experienced physicists, but also a much broader audience. In order to study the fundamental laws of physics and particle theories, Quantum Field Theories (QFTs) are required to be empirically verified by experimental methods. Due to the finite nature of such experiments, general predictions can only be verified up to the scale of the energy involved in these experiments. Hence, it is customary to divide all available mathematical theories into two categories: one which perfectly describes all empiric observations at low energies, and one which is a complete theory of the universe and includes a perfect description of high energy physics. The former class contains the infrared (IR) theories, and the latter all ultraviolet (UV) theories.

Due to technical energy limitations, it is impossible to construct experiments to verify UV complete theories up to infinitely large energy scales. Therefore, it is necessary to approximate these UV theories by another set of theories called Effective Field Theories (EFTs) in the low energy limit. Hence, by definition, all EFTs are in the IR range and correspond to the IR approximation of the underlying UV complete theory.

Not only does particle physics explain these mysteries of our universe, it also connects many of the most fundamental areas of physics in a much deeper way. Given their dependencies on energy scales, EFTs have a huge variety of applications across all sectors of research. In fact, every field in physics that uses one QFT approach or another - ie. any field that considers system with not conserved particle numbers - can (and does) exploit EFTs, especially when considering a specific physical scale these QFTs should be valid at. Hence EFTs are the perfect way of combining quantum theories with experiments, and phenomenology with experimental limitations.

I will demonstrate the utility of EFTs with an explicit example of lepton universality violation that have proved significant researched outcomes in recent years [1].

1.1 Example of EFTs in particle physics: lepton universality violation

In March 2021, reports analysing data from CERN [1] started to show relevant anomalies, highlighting possible discrepancies violating lepton universality in B meson decays, as seen in Fig. 3.2. The Standard model itself includes the electromagnetic force, whose

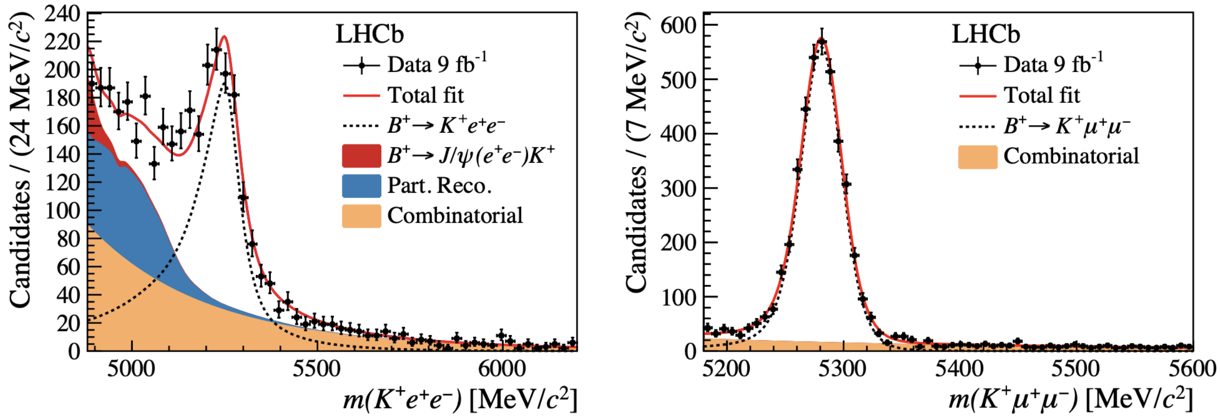

Figure 1.1: Reproduced from [1], showing the experimental evidence for unexplained differences between the electron and muon channels. The left plot shows the $B^+ \to K^+{+}e^+{+}e^-$ reaction, whilst the right plot visualises the results for $B^+ \to K^+{+}\mu^+{+}\mu^-$ Note how the peaks are slightly shifted, indicating different branching ratios between these channels.

quantum treatment of QED is described by a U(1) gauge group, the weak force being entailed with a SU(2) doublet as well as the strong force being described by SU(3) triplets in QCD. Naturally, three types of generations of matter were discovered in nature, all leptons and quarks come in three separate families, with their only difference being a difference in rest mass between generations whilst all other quantum numbers are shared across generations in the SM physics. It is for this feature that electrons, muons and taus ought to share similar reaction characteristics in scattering processes. Whilst particles with heavier masses generally correspond to shorter life times due to the uncertainty principle and scaling in the Feynman propagator [2], there is no obvious reason why such heavier masses should behave *in another way* differently compared to lighter particles of the same flavour. Nonetheless, physicists have still been running similar experiments to measure differences in branching ratios between muon and electron channels to infer wether an equivalent number of electrons and muons was produced. Such an equality between observed electron and muon numbers within the experimental errors could only mean that both flavours of this triplet [1] have to

[1] ignoring the tau where similar arguments could be made

couple to the weak force in the same way. Given that scattering cross sections are generally proportional to the corresponding coupling constants in the amplitudes and Lagrangian, this means the only way to accommodate for such observations is for electrons and muons to have the same coupling constants. Extending this to include taus, such a lepton flavour universality (LFU) has been implicitly assumed since the the early stages of the standard model.

However, it is clear that particle physics still cannot answer why there are only 3 generations (and if there are even more energetic ones), so this LFU has effectively been only an assumption that has not been disproven over the last decades.

It was therefore very exciting to see experimental reports [1] indicating an anomaly of 3.1 standard deviations in March 2021, unexplained by the SM, whose analysis concerned the electron and muon decay channels of B mesons. Charged B mesons decay into charged kaons and a lepton-antilepton pair [2], and the only allowed mediators in the SM for this is the photon, the $W^{\pm}$ and some heavy quarks [1]. These SM mediators would couple to the dileptons in the same way quantitatively. However, the analysis showed the branching ratios differed for the two channels and was not unity, implying there must be a fundamental reason that is beyond-the-standard-model (BSM), explaining this discrepancy. Of course it is possible that another BSM mediator coupled to each flavour differently, thus producing such an inequality without having to obey the SM rules as such a mediator would have to couple to both leptons and quarks at the same vertex[3]. These BSM gauge bosons are called leptoquarks and have been modelled to provide additional scattering processes[3]. Some of these are visualised in Fig. 1.2.

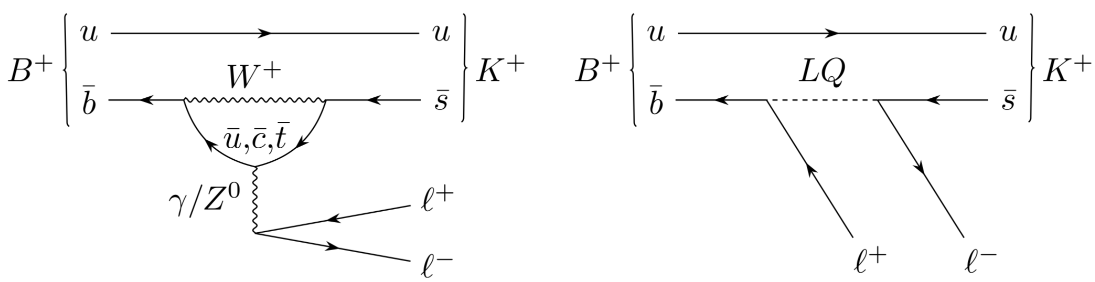

Figure 1.2: Reproduced from [1]. The left diagram shows Standard Model weak interactions in the charged B meson decay by exchanging charged W bosons, whilst the right diagram shows additional leptoquark mediators coupling directly to leptons and quarks.

Given the complexity of combining QCD with the electroweak sector, it is not suprising that a big range of possible operators to be included in the Lagrangian have to be considered, and using an EFT approach has helped. Promising leptoquark models obey $SU(3)_c \times SU(2)_L \times U(1)_Y$ symmetries [3] Then one can consider vector fields transforming under $SU(2)_L$, ie $SU(2)_L$ doublets, as well as scalar fields transforming under $U(1)_Y$, i.e. $U(1)_L$ singlets. These give appropriate representations for scalar and vector leptoquarks and some of their quantum numbers are listed in Fig. 1.3.

	S_1	$\mathbf{S}_3$	V_2	R_2	R'_2	U_1	$\mathbf{U}_3$
spin	0	0	1	0	0	1	1
$F = 3B + L$	-2	-2	-2	0	0	0	0
$SU(3)_c$	3^*	3^*	3^*	3	3	3	3
$SU(2)_L$	1	3	2	2	2	1	3
$U(1)_{Y=Q-T_3}$	1/3	1/3	5/6	7/6	1/6	2/3	2/3

Figure 1.3: Reproduced from [3], this table lists the quantum numbers and properties for relevant scalar and vector theories.

For instance, one of the terms added to the Lagrangian for the S_1 operator case with a U_1^μ vector leptoquark [3] is $L_{s_1} = h^{ij}_{LQ}\bar{u}_i\gamma_\mu P_L \nu_j U_1^\mu$, which is clearly not contained in the SM because it does not conserve lepton number at the vertex level. By considering all terms of this form, it can be shown that the Wilsonian coefficients $C^{ij}_{S_1}$ is proportional to $C^{q_i l_j}_{S_1} \propto \frac{h^{ij}_{LQ_1}}{G_F V_{qb} M^2_{LQ}}$ Once these coefficients are known, experiments can be conducted to test these theories, as in [3]. Some results are shown in 1.4 where the physical exclusion region allowed by EFTs is visualised and compared to experimental observations. How similar theoretical EFT bounds and constraints can be inferred shall be the main topic of this dissertation.

1.2 Introduction to EFT

In any case, studying such EFT descriptions enables a deeper understanding of experimental results, make predictions that are necessary for the unification of fundamental forces and connect different fields in physics that usually aren't closely related.

In practice such an EFT is often obtained by integrating out the heavier particle fields of the UV theory, so that only the lighter particle fields remain in the IR range, giving the EFT, and this method will be explained in more detail in later chapters.

In general, major collider experiments, such as the Large Hadron Collider at CERN, are only provided with limited energy, which implies an upper bound on the energy sector investigated by the collider's particle searches. Specifically for the LHC [4], particles have a centre-of-mass energy of $\sqrt{s} = 8$TeV, which is below the rest energy of a vast number of Beyond-the-Standard-Model (BSM) particles that are predicted. Hence, to test current predictions, it is required to narrow down the huge parameter

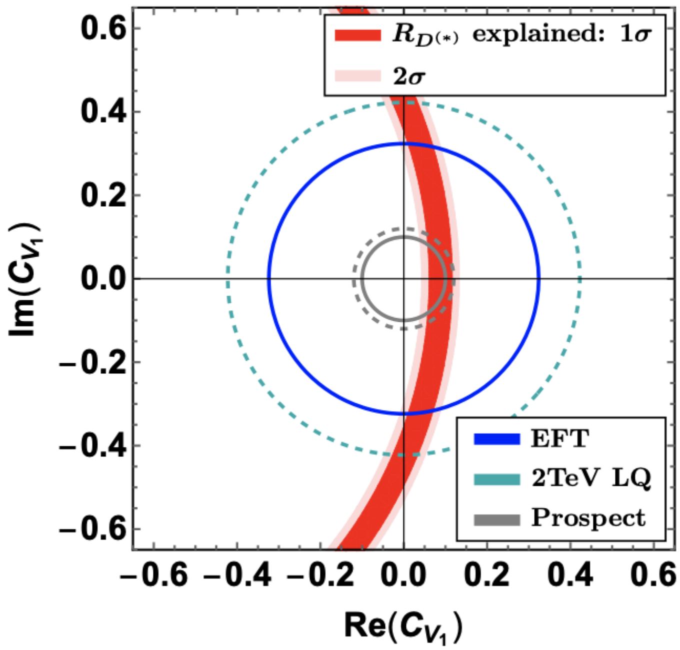

Figure 1.4: Reproduced from [3], this diagram demonstrates the theoretical EFT conditions, leading to an exclusion of all data outside the blue circle. The red area corresponds to the experimental observations, and combining both significantly narrows down the Wilsonian coefficients, i.e. the parameter space.

space associated with experiments to ensure experimental resources are used efficiently in searches for new physics.

As a result, this establishes the need for theoretical methods of narrowing down what energy sector to look at. Given that ordinary SM as well as BSM physics is described by EFTs[2], we therefore need to establish generalised bounds for the corresponding EFTs. Whilst there are many ways of obtaining bounds due to analyticity, unitarity etc, most bounds are associated with positivity of the corresponding parameters.

Whilst the concept of positivity conditions is not new, it has only recently lead to new results since the early 2000s, which is a relatively new approach of particle phenomenology. This is also due to the fact that until the late 20th century, the energy of particle colliders increased rapidly, producing a large number of particle discoveries, so that theoretical optimisation bounds were not strictly necessary.

As EFTs are so useful in many areas of physics, and their parameters being bounded by positivity bounds, this allows experiments in all of these fields of physics to search for new particles or interactions described by EFTs.

Typically investigating such interactions generally involves 2 processes: scattering

[2]e.g. with SMEFT approaches

and decays, though the latter can be thought of a special form of the former. Scattering processes always involve the interaction between n incoming and m outgoing particles, where the incoming and outgoing particles are allowed to change their quantum numbers (flavour, colour, charge etc) according to the dynamical constraints of the force causing the interaction. Scattering does not only include geometric scattering like particles bouncing off of a hard surface, but also simple repulsions and any general interaction with other particle fields. Hence the term scattering really describes the interaction between different quantum fields. This allows to use the standard QFT description of particle fields to associate the theoretical field interaction with experimentally observable scattering processes. Hence, it is required to find a relationship between the quantities measured in laboratories, and the theoretical predictions. Experimental particle physics entirely evolves around scattering cross sections, that describe an effective area being a measure of the interaction probability. In QFT, on the other hand, the interaction is described by an additional term in the Lagrangian, which pertubatively gives raise to Feynman diagrams describing the interaction / scattering. Clearly the interaction Lagrangian must be related to the scattering cross section, so establishing this relationship is the purpose of particle phenomenology. There actually exist numerous ways of calculating the scattering contribution of interaction Lagrangians, e.g. canonical pertubation theory, Feynman rules, or path integrals, however, these must all arrive at the same physical results (and do). Independently of which of these methods is used, the Feynman diagrams corresponds to a scattering matrix, or S-matrix, and each diagram results in a quantum mechanical scattering amplitude. Now all that is left to do is relate that scattering amplitude to the cross section, and we have a full relationship between QFTs and particle physics. The optical theorem describes such a relationship, and allows to express the differential cross section in terms of the scattering amplitude. In fact, as will later be shown, the optical theorem is a direct result of the unitarity condition our theories must have.

Also, the EFT positivity bounds usually involve only the scattering amplitude.

Hence we can identify the required steps to calculate positivity bounds for any given theory:

1. Investigate the Lagrangian and write down the scattering amplitude, using Feynman rules or otherwise

2. Apply the positivity bounds on the Wilsonian coefficients / coupling constants in the Lagrangian

3. Calculate the scattering amplitude, which already includes the positivity bounds

4. Calculate cross section from scattering amplitude using the optical theorem

Following this procedure allows the calculation of nearly every physical process described by IR physics, as long as the corresponding Lagrangian is a priori known. Of course, if it is not known a priori because eg a UV Lagrangian is available instead, this can be turned into an IR / EFT Lagrangian using a low energy approximation.

It should be noted that the following theories described in this dissertation are EFTs, valid in the low energy region and already renormalised. Introducing a cut-off scale Λ as a regulator in a QFT generally requires a distinction between 1) the IR physics, which can be described by the QFT and has to result in finite observables, and 2) the UV physics, which generally includes the very high energy range above Λ and therefore includes singularities and infinities that cannot be described by the QFT any more. Renormalising the QFT removes the regulator, and the singularities with it, so that the renormalised QFT is valid across all energies. Hence by assuming that our EFTs are already renormalised, we do not need to be concerned about the cut-off scale of possible renormalisation, which in general, might be different to the scale we set to be the boundary between IR and UV physics. A result of this is that we can only focus on the tree-level physics, as the renormalisation of vacuum expectation value (vev), mass, coupling constants etc is due to loops and quantum loop corrections. Hence assuming we already have renormalised QFTs / EFTs across this dissertation allows us to neglect all loop diagrams and calculations.

It also clear that one can distinguish known IR physics from the unknown UV physics by simply choosing the energy boundary to be the biggest experimentally available energy order of magnitude ie. 8 TeV or slightly above. However, since the experiments (hopefully) improve all the time, this also means that the boundary between these two regimes continuously shifts towards the unknown and a clear cut between IR and UV physics does not always exits, and is sometimes rather smeared out.

In the following I will rederive the basic theorems used in calculating the EFT constraints, including the unitarity and analyticity conditions. We will see that the unitarity constraint results in the optical theorem, and the analyticity one in the dispersion relation for the EFT. This is then followed by a calculation of explicit scalar bounds to implement the steps described above. I will then apply similar methods in deriving the vector bounds of the Proca EFT Lagrangian. Lastly, I will consider a subset of electroweak operators to calculate the positivity bounds for quartic vector gauge boson scattering that have implications for BSM physics.

Chapter 2

Basics

2.1 Scattering

2.1.1 QFT

Throughout this dissertation the metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.1}$$

is used since QFT approaches for the particle physics calculations were utilised and this convention is most frequently used in QFT / High Energy Physics. When comparing the numerical results to any papers using the other choice of metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, as used in research in general relativity, it should be taken into account that tensors with odd numbers of indices, i.e. tensors of odd rank, in the presented Lagrangian below will therefore have additional minus signs. Individual terms in the corresponding bounds thus may or may not have different signs, so taking care of the metric convention is crucial.

It is always possible to separate the Hamiltonian into a free part H_0^S and interacting part H_{int}^S so that $H^S = H_0^S + H_{int}^S$ holds. This then allows to define the free unitary evolution operator

$$U_0(t) = e^{-itH_0^S}. \tag{2.2}$$

The free Hamiltonian is then conserved as

$$H_0^I = U_0^{-1}(t)H_0^S U_0(t) = U_0^{-1}U_0 H_0^S = H_0^S.$$

For the interacting part H_0^S and H_{int}^S do not commute any more, implying

$$H_{int}^I = U_0^{-1}(t)H_{int}^S U_0(t) \neq H_{int}^S$$

Hence it is sensible to focus on the evolution operator in the interacting picture, describing the time dependence of the states as

$$|\Psi(t_2)\rangle^I = \hat{U}^I(t_2,t_1)\,|\Psi(t_1)\rangle^I$$

with

$$\hat{U}^I(t_2,t_1) = e^{-iH^I_{int}(t)t}.$$

This operator relation between $\hat{U}^I(t,t_0)$ and $\hat{H}^I_{int}$ is equivalently expressed as a differential equation

$$\frac{\partial \hat{U}^I(t,t_0)}{\partial t} = -i\hat{H}^I_{int}\hat{U}^I(t,t_0),$$

which can now be recursively solved to get

$$\hat{U}^I(t,t_0) = \mathbb{1} + (-i)\int_{t_0}^{t} \mathrm{d}t_1 H^I_{int}(t_1) + (-i)^2\int_{t_0}^{t}\int_{t_0}^{t} \mathrm{d}t_1\mathrm{d}t_2 H^I_{int}(t_1)H^I_{int}(t_2) + ...$$

or of course one can expand $\hat{U}^I(t_2,t_1) = e^{-iH^I_{int}(t)t}$ directly to get the same result, although with this direct method it is more difficult to see which form the integration limits are. Now reorganising this expression by introducing time ordering T ensures causality and one arrives at the Dyson expansion

$$\hat{U}^I(t,t_0) = \frac{\mathbb{1}}{0!}+\frac{(-i)}{1!}\int_{t_0}^{t} \mathrm{d}t_1 H^I_{int}(t_1)+\frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t} \mathrm{d}t_1\mathrm{d}t_2 T\{H^I_{int}(t_1)H^I_{int}(t_2)\}+... \quad (2.3)$$

This is clearly a pertubative expansion that can be approximated by truncating the infinite sum at any given order. This is the method used for calculating evolution operators throughout this dissertation, and the interaction picture will be implicitly assumed.

Consider an initial state $|i\rangle$. An evolution operator will then transform it into a final state $|f\rangle = \hat{U}^I(t_f,t_i)\,|i\rangle$ and the probability of measuring the final state is then the modulus squared of

$$M = \langle f|f\rangle = \langle f|\hat{U}^I(t_f,t_i)|i\rangle$$

Now letting the initial and final states exist at $t = \pm\infty$ so that the plane wave approximation is valid allows to define the scattering amplitude S and scattering matrix or S-matrix $\hat{S}$:

$$S = \lim_{T\to\infty} \langle f|_T\,\hat{S}\,|i\rangle_{-T}$$

Of course one can think of the S-matrix as giving an indication of the transfer from $i \to f$, however, quantitatively this is the same as just directly measuring the probability of obtaining a final state.

By including time ordering as well as normal ordering[1], the scattering amplitude can then be obtained by calculating the Fourier transforms of the n-point functions

$$S = \lim_{T\to\infty} \langle q_1...q_n|_T\,\hat{S}\,|p_1...p_m\rangle_{-T} \quad (2.4)$$

[1]to remove some simple singularities, more complicated divergencies will have to be treated with regularisation and renormalisation

$$= \lim_{t\to\infty} \prod_{i=1}^{n} \prod_{j=1}^{n} 4E_i E_j \int \int d^3x_i d^3y_j e^{ipx} e^{-iqy}$$

$$\times \langle\Omega|_t : \phi(x_1)...\phi(x_n) :: Te^{-i\int_{t_0}^{t} \text{ṭ}' H_{int}(t')} :: \phi(y_1)...\phi(y_m) : |\Omega\rangle_{-t}$$

This correlator now allows to utilise Wick's theorem and write down the Feynman rules that immediately give the LHS, i.e the scattering amplitude S, in terms of the interaction Hamiltonian, $\hat{H}_{int}$ on the RHS, of a theory.

Derivative couplings

We quantise scalar fields by promoting ϕ to $\hat{\phi}$ and writing this in terms of creation and annihilation operators:

$$\hat{\phi}(x) = \int \frac{dp^4}{(2\pi)^4}(\hat{a}(p)e^{-ip\cdot x} + \hat{a}^\dagger(p)e^{ip\cdot x}) \tag{2.5}$$

Now it is apparent that when a specific vertex is analysed, incoming particles get destroyed at the vertex and outgoing particles get created. This means that the incoming particles must be in the annihilation mode $\hat{\phi}(x) = \int \frac{dp^4}{(2\pi)^4}\hat{a}(p)e^{-ip\cdot x}$ and the outgoing particles in the creation mode $\hat{\phi}(x) = \int \frac{dp^4}{(2\pi)^4}\hat{a}^\dagger(p)e^{ip\cdot x}$ *at the vertex*. Away from the vertex at x, i.e. without the interaction and in free space, $\hat{\phi}$ is the superposition of both modes again and particle number is conserved. At the vertex, the particle number is not constant, so one specific operation and mode gets picked out. Similarly, at the vertex the momenta of the particles are known, and one can at least express every momentum in terms of the other momenta due to momentum conservation. This means that an integration over all 4-momenta is not necessary as the incoming and outgoing particles only have one specific momentum each. Now imagine the Lagrangian included terms proportional to $\partial_\mu\phi$, then this implies that for incoming particles

$$\partial_\mu\phi = \partial_\mu\hat{a}(p)e^{-ip\cdot x} = -ip_\mu\hat{a}(p)e^{-ip\cdot x} = -ip_\mu\hat{\phi}$$

and for outgoing particles

$$\partial_\mu\phi = \partial_\mu\hat{a}^\dagger(p)e^{ip\cdot x} = ip_\mu\hat{a}^\dagger(p)e^{ip\cdot x} = ip_\mu\hat{\phi}.$$

In fact, for every derivative acting on fields in the Lagrangian, the derivative can therefore be directly replaced by a momentum factor $\pm ip_\mu$ in the scattering amplitude. Here the sign is important and distinguishes an incoming particle $(-p_\mu)$ from an outgoing one $(+p_\mu)$. Vector fields are quantised in a similar manner, although have the added complication that in addition to the creation and annihilation operators and complex phase Fourier factors, each mode also has an additional vector factor that entails the representation of the field in the four spacetime dimensions. This vector factor is the polarisation ϵ_μ, which may or may not be time-dependent i.e. non-constant. As a result, (massless) vector fields are quantised as

$$\hat{\phi}(x) = \int \frac{dp^4}{(2\pi)^4} \Sigma_{i=1,2}\epsilon_i \big(\hat{a}(p)e^{-ip\cdot x} + \hat{a}^\dagger(p)e^{ip\cdot x}\big) \tag{2.6}$$

Here the total polarisation ϵ is given by the sum over the two transverse and the longitudinal/scalar polarisation basis vectors ϵ_i. However, in Lorentz gauge, $\partial_\mu A^\mu = 0$, and after removing any residual gauge freedom, only two physical degrees of freedom remain. These correspond to the choice of the two transverse polarisation bases, with the longitudinal mode not contributing to any physical processes any more. Hence in the quantisation the sum only sums over $i = 1$ and $i = 2$. Note this is for massless vectors only and massive vectors will be dealt with in chapter 4.

Hence any derivative terms in the Lagrangian, such as a rank-2 tensor $\partial_\mu A^\nu$ or a rank-0 tensor $\partial_\mu A^\mu$, will result in similar, additional momentum pre-factors in the scattering amplitude as above for the scalar case. However, given that the polarisation is a vector itself too, there will now also be contractions between polarisations and momenta, and between polarisations-polarisations, requiring a complicated treatment that will be derived for massive vector fields in chapter 4.

2.1.2 Mandelstam variables

The momentum conservation for $1 + 2 \to 3 + 4$ scattering is explicitly given as

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$$

This is usually implemented by including a delta function such as $\delta(\Sigma p^\mu) = \delta(p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu)$ in all Feynman terms that when integrated over at least one momentum ensures this conservation law. Also the on-shell condition for free particles and external legs is

$$p_i^2 = m_i^2$$

for $i \in \{1, 2, 3, 4\}$.This is the second general condition scattered particles have to satisfy. Note that for particles on internal lines the particles are off-shell, so only the first conservation law has to be fulfilled, which is done be ensuring momentum conservation at each vertex separately. When analysing the tree-level diagrams of cubic interactions, 3 specific combinations of momentum contractions appear so frequently that the following definitions are commonly[2] used:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_4 - p_2)^2$$

$$u = (p_1 - p_4)^2 = (p_3 - p_2)^2$$

[2]these variables would have the opposite sign for (-+++)

Adding all 3, it can easily be verified that

$$s + t + u = \Sigma_{i=1}^{4} p_i^2 = \Sigma_{i=1}^{4} m_i^2 = 4m^2$$

In the following $\vec{p} \circ \vec{q}$ is taken for 3-vector dot products and $(p \cdot q) = p_\mu \, q^\mu$ for denoting 4-vector contractions. In the ultra-relativistic limit $E >> m_0$, so $p \cdot p = E^2 - \vec{p} \circ \vec{p} = m^2$ implies $E^2 \approx \vec{p} \circ \vec{p}$. Hence these variables simplify to

$$s = 2m^2 + 2p_1 \cdot p_2 \approx 2p_1 \cdot p_2,$$

$$t = 2m^2 - 2p_1 \cdot p_3 \approx -2p_1 \cdot p_3$$

and

$$u = 2m^2 - 2p_1 \cdot p_4 \approx -2p_1 \cdot p_4.$$

This is also known as the Eikonal approximation. Using the equations above, any two momentum contractions can be written as

$$p_1 \cdot p_2 = p_3 \cdot p_4 = \frac{s - 2m^2}{2},$$

$$p_1 \cdot p_3 = p_2 \cdot p_4 = \frac{2m^2 - t}{2}$$

and

$$p_1 \cdot p_4 = p_2 \cdot p_3 = \frac{2m^2 - u}{2}.$$

This is particularly useful because it allows for any two momentum contractions of *different* momenta to be expressed in terms of Mandelstam variables. Additionally, this shows that due to momentum conservation there is a redundancy here since we can reduce the initially 6 possible contractions to only 3 different combinations, simplifying calculations significantly.

2.1.3 scattering kinematics

It is important to distinguish between elastic/inelastic collisions and elastic / inelastic scattering. General scattering is clearly related to particles colliding, however, both can be *kinematically* different. Collisions refer to the effects on the whole system, whereas scattering considers the effects on individual particles. We hence have to distinguish between four cases:

1. Inelastic collisions conserve the total energy of the system, but the total kinetic energy is not conserved. The energy loss is converted into heat in classical mechanics, which in relativity corresponds to increased rest mass.

2. Elastic collision conserve the total kinetic energy in the system, but not the kinetic energy of each particle individually. There is no heat so no mass increase.

3. Elastic scattering conserves the individual kinetic energies (and therefore the total kinetic energy in the system as well)

4. Inelastic scattering does not conserve individual kinetic energies. This may or may not lead to conservation of the total kinetic energy.

It is apparent that all elastic scattering is due to elastic collisions, but not all elastic collisions cause elastic scattering. For instance, both Compton scattering and Newton's pendulum are examples of elastic collisions that are inelastic scattering.

Requiring elastic scattering by choosing a constant kinetic energy T, however, is not the strictest requirement possible for such processes. This is because both classically ($T = \frac{1}{2}m|\vec{v}|^2$) and relativistically ($T = (\gamma - 1)mc^2 = (1 - \frac{|\vec{v}|^2}{c^2})^{-\frac{1}{2}} - 1)mc^2$) invariant T only implies *constant speed*, with no restriction on the velocity direction. Therefore one can impose an even stronger limitation, requiring the velocity direction *and* magnitude to be invariant with respect to the interaction. This is additional constraint is known as *forward scattering limit* and is equivalent to requiring the Mandelstam variable $t = 0$. As a result, the initial and final states can only be identical ($|i\rangle = |f\rangle$) if and only if both elastic scattering and forward scattering limited are assumed.

This allows to infer that in the forward limit, i.e. $|i\rangle = |f\rangle$ the scattering angle ϕ is zero in this approximation, $t = 0$ for $\phi = 0$.

Scattering dynamics versus kinematics

Another aspect to note is that there are two types of conditions that have to be satisfied in order to get physical scattering processes. Firstly, the kinematics described above, i.e. momentum and energy conservation, has to be fulfilled. Secondly, the the dynamics of the system has to be consistent and is described by the equations of motions, i.e. the Euler-Lagrange-equations, both classically and in QFT prescriptions. The latter (the dynamics) would be inferred from the specific form of the Lagrangian, whereas the kinematics would be *a priori* known once the fundamental symmetries of the system are identified (e.g. Poincaré invariance). Considering both dynamics and kinematics give additional constraints, and their combination have very significant implications for scattering. Suppose we consider a set of particles described by the simple Klein-Gordon-Lagrangian for ϕ^4 theory

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{2.7}$$

and would like our theory to include inelastic collisions too. In general, real empirically observable processes will always lead to partial energy loss via heat generation and subsequent exchange with the environment. Therefore, we expect heat to be included in the kinetic terms $T = (1-\gamma)mc^2$ of the outgoing particles, which then must have smaller rest mass to compensate their increased thermal energy. However, for the considered ϕ^4 model this is impossible because the Lagrangian only has one single species with one specific mass m. Therefore ϕ^4 theory can only result in elastic scattering, showing the importance of distinguishing the kinematics from the dynamics of a given Lagrangian. In chapter 3 and 4 similar, albeit more complex, Lagrangians with one single mass term will be analysed, so for these models elastic collisions is a necessary condition. Chapter 5's treatment of the Standard Model and SMEFTs however include a vast number of different possible mass terms, so omitting inelastic collisions for SMEFTs is in fact an optional assumption instead as its omission is not implied by the field dynamics of the SMEFT Lagrangians.

2.1.4 scattering cross sections

We start by defining the scattering cross section following arguments made in. Recalling that the particle flux $\Phi = nv$ with number density $n = \frac{N'}{V}$ and particle speed v, the scattering cross section is defined as

$$\sigma = \frac{1}{\Phi}\frac{N}{T},$$

where N is the number of scattered particles per unit time T. The total number of particles in the beam is N' however, so clearly $N \leq N'$, and the scattering probability is exactly $P_S = \frac{N}{N'}$. Rewriting the flux density using average speed $v = \frac{L}{T}$ gives $\Phi = \frac{N'}{V}\frac{L}{T}$ and inserting this expression into the cross section yields

$$\sigma = \frac{N}{N'}\frac{V}{L} = P_S\tilde{A},$$

where $\tilde{A} = \frac{V}{L}$ is the average area. Hence from this expression it is apparent that using this definition of the cross section above, σ can be interpreted as the effective area corresponding to the scattering / interaction probability.

Assuming N' =constant, we have $\mathrm{d}P_S = \frac{\mathrm{d}N}{N'}$, so

$$\mathrm{d}\sigma = \frac{N'\mathrm{d}P_S}{T\Phi}$$

or for 1 particle beams with $N' = 1$

$$\mathrm{d}\sigma = \frac{\mathrm{d}P_S}{T\Phi}$$

The cross section is independent of the experimental parameters and only depend on the underlying physical scattering process. If one wishes to include experimental dependency, it is common to rewrite this in terms of the integrated luminosity $L = \int \mathcal{L}\mathrm{d}t$, which is dependent on the explicit parameters. The integrated luminosity is defined as $L = \frac{\mathrm{d}N}{\mathrm{d}\sigma}$, yielding

$$\mathrm{d}N = \frac{L}{T\Phi}\mathrm{d}P_S.$$

(For a one particle beam the particle flux is clearly given by $\Phi = \frac{|\vec{v}|}{V}$) Assuming only 2 colliding particles, we only have 2 initial states, so the incoming beam only has 2 particles[3]. In the Centre-of-Mass (CoM) frame[4] this corresponds to $\Phi = \frac{|\vec{v_2}-\vec{v_1}|}{V}$. Hence

$$\mathrm{d}\sigma = \frac{V}{T}\frac{\mathrm{d}P_S}{|\vec{v_2}-\vec{v_1}|}.$$

In quantum mechanics the probability amplitude A of transition processes is given by

$$A = \frac{\langle f|S|i\rangle}{\sqrt{\langle f|f\rangle \langle i|i\rangle}},$$

where f is the final state and i is the initial state. Here we consider 2-n scattering. The same form holds in QFT for scattering processes. Thus the probability is

$$P_S = |A|^2 = \frac{\langle f|S|i\rangle \ \langle i|S|f\rangle}{\sqrt{\langle f|f\rangle \langle i|i\rangle}^2} = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle}$$

Now $|i\rangle = |i_1 i_2\rangle$ and $|f\rangle = \bigotimes_{j\in f}|f_j\rangle$ are asymptotic states, that is they correspond to the system for $t \to \infty$ and $t \to -\infty$. Hence they are constant as the interaction will be over at $t = \infty$ and won't have started to affect states at $t = -\infty$. Taking the differential of this then results in

$$\mathrm{d}P_S = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle}\prod_{j\in f}\frac{V}{(2\pi)^3}\mathrm{d}^3 p_j$$

for $p_j = \frac{2\pi i}{L}$ and taking care to only multiply the final states.

The usual quantisation of momentum states results in $\langle p|p\rangle = 2E_p(2\pi)^3\delta^3(0)$. We can rewrite $\delta^3(0) = \frac{1}{(2\pi)3}\int \mathrm{d}^3x = \frac{V}{(2\pi)^3}$, where V is clearly the infinitely big space. Similarly $\delta^4(0) = \frac{TV}{(2\pi)^4}$

Now expressing the particle state normalisation as

$$\langle p|p\rangle = 2E_pV,$$

as in [5] one therefore has

$$\langle i|i\rangle = \Big(\langle i_1|\,\langle i_2|\Big)\Big(|i_1\rangle\,|i_2\rangle\Big) = \langle i_1|i_1\rangle \langle i_2|i_2\rangle = 4E_1V^2E_2$$

[3]We only deal with 2-2 scattering because the fields get stronger the closer the particles are to each other, and it is nearly impossible for 3 or more particles to happen to be at exaclty the same distance to each other. Hence the probability of all of them interacting at the same time is nearly zero, and 2-2 processes dominate. Of course in beams with millions of particles thousands of these 2-2 scatterings happen, but they are all treated either independently of each other, or after each other. The essential point is that 2-3 or 3-3 virtually never happen.

[4]Note that in the lab frame we have a 1 particle beam with speed v_1, and in CoM frame we have 2 particles each with speed $\frac{|\vec{v}_1-\vec{v}_2|}{2}$, so $\Phi_{\text{CoM}} = \frac{2\cdot|\vec{v_2}-\vec{v_1}|}{V\cdot 2} = \frac{|\vec{v_2}-\vec{v_1}|}{V}$, and this particle flux is consistent in either frame

and

$$\langle f|f\rangle = \left(\bigotimes_{j\in f}\langle f_j|\right)\left(\bigotimes_{j\in f}|f_j\rangle\right) = \prod_{j\in f} 2E_j V.$$

Writing the scattering matrix as $S = \mathbb{1} + T$, where T is the non-trivial transfer matrix and 1 corresponds to the trivial scattering (free propagation and no interaction), then neglecting the trivial part we can define the scattering amplitude $\langle f|T|i\rangle = (2\pi)^4\delta^4(\Sigma p)$. The delta function here is due to the trivial part, since this would result in momentum conservation. Hence adding the delta function ensures this conservation law and entails everything we can infer from the free propagation. [5] Squaring[6] this

$$|\langle f|T|i\rangle|^2 = (2\pi)^8\delta^4(0)\delta^4(\Sigma p)|M|^2 = TV(2\pi)^4\delta^4(\Sigma p)|M|^2$$

Putting all together into equation:

$$\mathrm{d}\sigma = \frac{V}{T}\frac{1}{|\vec{v_2}-\vec{v_1}|}\frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}\prod_{j\in f}\frac{V}{(2\pi)^3}\mathrm{d}^3p_j$$

$$= \frac{V}{T}\frac{1}{|\vec{v_2}-\vec{v_1}|}\frac{TV(2\pi)^4\delta^4(\Sigma p)|M|^2}{4E_1E_2V^2\prod_j 2E_jV}\prod_{j\in f}\frac{V}{(2\pi)^3}\mathrm{d}^3p_j$$

$$= \frac{1}{|\vec{v_2}-\vec{v_1}|}\frac{(2\pi)^4\delta^4(\Sigma p)|M|^2}{4E_1E_2\prod_j 2E_j}\prod_{j\in f}\frac{1}{(2\pi)^3}\mathrm{d}^3p_j$$

$$= \frac{|M|^2}{4E_1E_2|\vec{v_2}-\vec{v_1}|}\left(\frac{(2\pi)^4\delta^4(\Sigma p)}{\prod_j 2E_j}\prod_{j\in f}\frac{1}{(2\pi)^3}\mathrm{d}^3p_j\right)$$

$$= \frac{|M|^2}{4E_1E_2|\vec{v_2}-\vec{v_1}|}\mathrm{d}\Pi_{\mathrm{LIPS}},$$

where the Lorentz invariant phase space factor is $\Pi_{\mathrm{LIPS}} = \frac{(2\pi)^4\delta^4(\Sigma p)}{\prod_j 2E_j}\prod_{j\in f}\frac{1}{(2\pi)^3}\mathrm{d}^3p_j$ Of course we set V, T from the cross section definition equal to the infinitely big V, T of the QFT norms. However, if we have a 1 particle beam then clearly one can define it to cover all space and lasting for infinitely long T until the interaction. Hence we can cancel the V and T.

2-2 scattering

Taking the special case of 2-2 scattering, the CoM frame the energy is conserved according to $E_{CoM,before} = E_1 + E_2 = E_3 + E_4 = E_{CoM,after}$. Also $\vec{p_1} = -\vec{p_2}$ and $\vec{p_3} = -\vec{p_4}$, due

[5] Note that this scattering amplitude is not the same as the probability amplitude A above because A was normalised by dividing by ii and ff, whereas M is not normalised. However, the infinitely big normalisation cancels soon, so this difference does not matter here

[6] Note $\delta^8(\Sigma p) = \delta^4(0)\delta^4(\Sigma p)$

to momentum conservation, so we can define $p_f = |p_3| = |\vec{p}_4|$. We take $E_{CoM} = E_1 + E_2$ for now and then show that our calculations must also imply $E_3 + E_4 = E_C oM$ The Lorentzian phase space becomes

$$\mathrm{d}\Pi_{\mathrm{LIPS}}(p_3, p_4) = \frac{\delta^4(\Sigma p)}{16\pi^2 E_3 E_4} \mathrm{d}^3 p_3 \mathrm{d}^3 p_4 \tag{2.8}$$

Writing out the abbreviation $\delta^4(\Sigma p_\mu) = \delta^4(\Sigma p_\mu^i - \Sigma p_\mu^f) = \delta\left(\Sigma E^i - \Sigma E^f\right)\delta^3\left(\Sigma \vec{p}^i - \Sigma \vec{p}^f\right)$ then after integrating with respect to the spatial $\mathrm{d}^4 p_3$ eqn becomes

$$\mathrm{d}\Pi_{\mathrm{LIPS}}(p_3) = \int_{p_4} \mathrm{d}\Pi_{\mathrm{LIPS}}(p_3, p_4) = \frac{\delta(E_3 + E_4 - E_{\mathrm{CoM}})}{16\pi^2 E_3 E_4} \mathrm{d}^3 p_3 \tag{2.9}$$

or in spherical coordinates

$$\mathrm{d}\Pi_{\mathrm{LIPS}}(p_3) = \frac{\delta(E_3 + E_4 - E_{\mathrm{CoM}})}{16\pi^2 E_3 E_4} \mathrm{d}\Omega p_3^2 \mathrm{d}p_3 \tag{2.10}$$

Integrating out the radial momentum so that only the differential angular dependence remains, we obtain

$$\mathrm{d}\Pi_{\mathrm{LIPS}}(\Omega_3) = \mathrm{d}\Omega \int \frac{\delta(E_3 + E_4 - E_{\mathrm{CoM}})}{16\pi^2 E_3 E_4} p_f^2 \mathrm{d}p_f \tag{2.11}$$

In order to compute this integral, the energies have to be expressed in terms of the momenta, via $E_3 = \sqrt{p_f^2 + m_3^2}$ and $E_4 = \sqrt{p_f^2 + m_4^2}$.

The argument of the delta function justifies the change in variables according to $y = E_3 + E_4 - E_{CoM}$, which implies

$$\frac{\mathrm{d}y}{\mathrm{d}p_f} = \frac{\mathrm{d}}{\mathrm{d}p_f}(E_3 + E_4 - E_{CoM}) = \frac{\mathrm{d}E_3}{dp_f} + \frac{\mathrm{d}E_4}{\mathrm{d}p_f}$$

$$= \frac{p_f}{\sqrt{p_f^2 + m_3^2}} + \frac{p_f}{\sqrt{p_f^2 + m_4^2}} = \frac{p_f}{E_3} + \frac{p_f}{E_4} = p_f \frac{E_3 + E_4}{E_3 E_4} = p_f \frac{E_{CoM}}{E_3 E_4}$$

so

$$\begin{aligned} \mathrm{d}\Pi_{\mathrm{LIPS}}(\Omega_3) &= \mathrm{d}\Omega \int_{m_3 + m_4 - E_{CoM}}^{\infty} \frac{\delta(0) p_f}{16\pi^2 E_{CoM}} \mathrm{d}y \\ &= \mathrm{d}\Omega \frac{\theta(E_{CoM} - m_3 - m_4) p_f}{16\pi^2 E_{CoM}} \end{aligned}$$

For the initial momenta we have $p_i = |\vec{p}_1| = |\vec{p}_2|$, so one can rewrite [5]

$$|\vec{v}_1 - \vec{v}_2| = p_i \frac{E_{CoM}}{E_1 E_2}$$

which allows to infer that the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 E_{\mathrm{CoM}}^2}\frac{p_f}{p_i}|M|^2\theta(E_{CoM} - m_3 - m_4) \tag{2.12}$$

This holds for any masses, so in the special case of elastic scattering, where all masses are equal, we obtain $p_f = p_i$ and $E_{CoM} - 2m \geq 0$ and thus

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 E_{\mathrm{CoM}}^2}|M|^2 \tag{2.13}$$

Here masses means rest masses. This clearly has kinematic implications, as opposed to the dynamic constraints due to the terms in the Lagrangian. For instance, if there is only one field in the Lagrangian, then clearly this field can only self-interact and all particles stem from the same Lagrangian, so the rest-masses of the scattered particles are all the same. If there is another field with a mass term of the same mass, then equation 2.13 can still be used. However, should the Lagrangian contain fields that have different mass terms, then this directly implies that the scattering is not elastic as the rest masses changes. Connecting these dynamic constraints to the kinematic scattering considerations allows us to infer the form of the cross section, depending on which theory is described within the Lagrangian[7].

2.2 EFTs

As mentioned before, EFTs can be thought of as set of valid theories in the low energy limit, below a certain cut-off scale Λ that separates the corresponding UV from the IR region. Heavy particles are particles with mass $m > \Lambda$ and therefore only appear in the corresponding UV complete theory (if such a UV completion is possible at all). Light particles with mass $m < \Lambda$ do appear in both regions and hence cannot be ignored in either. This distinction then allows two methods of dealing with EFTs. as described in [6]:

1. One can start with an appropriate UV complete theory, that describes the full physics picture including heavy masses, and then focus on the low energy range where the light particles dominate. Hence with this method the EFT is simply an approximation of the UV theory. This is known as the top-bottom formalism.

2. Irrespective of whether a UV theory exists, one can start in the low energy regime in the first place, determine the particle spectrum (which will consist of light particles only) and can then try to find a UV completion without knowing about the heavy particles *a priori*. This is known as bottom-up.

[7]In classical mechanics, the Euler Lagrange equations represent a generalised form of Newton's second law, describing the dynamics, and the work-energy theorem allows to connect the kinematics with the dynamics. One could argue that the way above is the QFT equivalent of a work-energy theorem

In any case, the EFT can be expressed as a sum over all operators $\hat{O}$, valid up to Λ, where this sum is parameterised by constant Wilsonian coefficients c_i:

$$L_{EFT} = \Sigma_i c_i \hat{O}_i \tag{2.14}$$

Consider a toy theory like in [6]

$$L_{UV} = \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}M^2\chi^2 + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3\chi,$$

where χ are the heavy fields with mass M and ϕ are the light fields with mass m, with $M > \Lambda > m$ by definition.

Then this Lagrangian must be part of a UV complete theory because it describes the full dynamics of the system. Now imagine one wanted to test this theory experimentally, or infer any other theoretical low energy characteristics, then the IR dynamics would not be immediately visible in this form of a UV complete theory since the heavy and light field χ and ϕ are coupled via $\hat{H}^I_{int} = \frac{\lambda}{3!}\phi^3\chi$. Hence one way of separating them is expressing ϕ in terms of χ and plugging this in back into L_{UV}. Calculating the Euler-Lagrange-equations, i.e. the equations of motion, for χ:

$$0 = \frac{\partial L_{UV}}{\partial\chi} - \partial_\mu \frac{\partial L_{UV}}{\partial\partial_\mu\chi} = -M^2\chi - \frac{\lambda}{3!}\phi^3 - \Box\chi = -(\Box + M^2)\chi - \frac{\lambda}{6}\phi^3$$

This then allows to rewrite this as

$$\chi = -\frac{\lambda}{6}\frac{\phi^3}{\Box + M^2},$$

that is the dynamics of χ is written as a function of the dynamics of ϕ. Plugging this back into L_{UV} now yields

$$L_{EFT} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda^2 M^2}{72}(\Box+M^2)^{-2}\phi^6 + \frac{\lambda^2}{72}\Big(\partial\big((\Box+M^2)^{-2}\phi^3\big)\Big)^2 - \frac{\lambda}{36}(\Box+M^2)^{-2}\phi^6$$

$$= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda^2 M^2}{72}\hat{O}\phi^6 + \frac{\lambda^2}{72}\big(\partial(\hat{O}\phi^3)\big)^2 - \frac{\lambda}{36}\hat{O}\phi^6,$$

where the operator $\hat{O}$ is defined as the differential operation $\hat{O} = (\Box+M^2)^{-2}$. Note that the label of the Lagrangian is now L_{EFT} because this Lagrangian now only contains light fields, so is not UV complete any more once this substitution has been made. Given that both in the Dyson expansion of canonical QFT as well as in the path integral formalism there will be some spacetime or momentum integrals of $\hat{H}^I_{int}$, so the fact that χ has now disappeared entirely in L_{EFT} and the *effective version* of H_{int} is why this method is also known as integrating out the heavy fields. L_{EFT} gives the full low energy dynamics of the system and it is apparent that it now contains ϕ^6 terms whilst in the UV Lagrangian only ϕ^3 terms, and along with χ, at most quartic vertices appeared. These quartic UV vertices now appear as sextic EFT vertices. In addition,

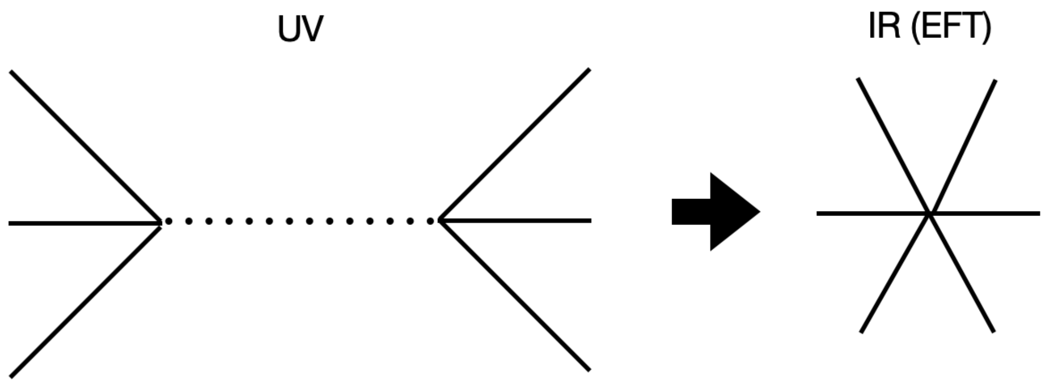

Figure 2.1: This sketch shows the relationship between UV and IR limits of the considered toy theory and allows to get an understanding of how these limits affect the order of the vertices.

the coupling constant of the sextic self-interaction of ϕ now has an effective coupling constant $\lambda' = \lambda^2$. This implies that e.g. any EFT Feynman diagrams up to the order of only one vertex must correspond to UV Feynamn diagrams up to the order of *two* vertices as in Fig. 2.1.

Hence by integrating out the heavy fields, those got absorbed into the lighter fields and changed their dynamics, self-interaction and coupling, so even though χ is not visible any more, it still affects the EFT in a hidden way.

Of course this argument also allows to approach any EFT that we know *a priori* by asking whether there is heavier field that is not included in the Lagrangian (because e.g. it is too heavy to be detected) and then finding out whether it is possible to conduct a UV completion to find and separate out the heavier fields again.

In summary, it is possible to switch between UV and EFT pictures by integrating out all heavy fields[6]. Which fields are included in the EFT is determined by choosing a cut-off scale Λ [7][8][9] so that all particles with mass $m > \Lambda$ are integrated out and discarded, so choosing Λ carefully for any given theory allows to study any specific particle spectrum of a UV complete theory more closely.

2.3 Bounds and constraints

2.3.1 Unitarity

One can derive the simplified optical theorem even more generally using the concept of unitary operators in QFT. In fact, one can even derive a new set of conditions due to unitarity, as in [5] and [6] and then show that this constraint directly implies the optical theorem from *unitarity*. Note that this holds in general, but the operator for our scattering consideration that we require to be unitary now is clearly the $\hat{S}$ matrix. In both QM and QFT a state transforms under an operator $\hat{A}$:

$$|i\rangle \to |i'\rangle = \hat{A}\,|i\rangle$$

and

$$\langle f| \to \langle f'| = \langle f|\,\hat{A}^\dagger$$

Hence it is immediately apparent that

$$\langle f|i\rangle \to \langle f'|i'\rangle = \langle f|\hat{A}^\dagger \hat{A}|i\rangle$$

Now due to conservation of probability as well as the probability's nature of transformation invariance the expectation values must be the same, i.e.

$$\langle f|i\rangle = \langle f|\hat{A}^\dagger \hat{A}|i\rangle$$

As this must hold for any f, i, it directly follows that

$$\hat{A}^\dagger \hat{A} = \mathbb{1},$$

i.e. that $\hat{A}$ is unitary.

For scattering the $\hat{S}$ matrix operator leaves the initial and final states invariant, that is it transforms the initial and final states under scattering in the trivial way $\langle f'| = \langle f|$ and $|i'\rangle = |i\rangle$. Hence the expression above directly becomes

$$\langle f|i\rangle \to \langle f|i\rangle = \langle f|\hat{S}^\dagger \hat{S}|i\rangle$$

and we need not even invoke conservation of probability[8] In any case $\hat{S}$ has to be unitary in order to qualify for a scattering operator.

Writing the S matrix as $\hat{S} = \mathbb{1} + i\hat{T}$ and neglecting the unscattered contribution, the T matrix elements expressed in terms of the scattering amplitude is $\langle f|T|i\rangle = (2\pi)^4\delta^4(p_i - p_f)M_{i\to f}$

Hence we can infer

$$\begin{aligned} \mathbb{1} = \hat{S}^\dagger\hat{S} &= (\mathbb{1} - i\hat{T}^\dagger)(\mathbb{1} + i\hat{T}) \\ &= \mathbb{1} - i\hat{T}^\dagger + i\hat{T} + \hat{T}^\dagger\hat{T} \end{aligned}$$

This implies

$$\hat{T}^\dagger\hat{T} = i(\hat{T}^\dagger - \hat{T}) \neq \mathbb{1}$$

[8] in some sense scattering seems to have an even stronger conservation of probability as it seems to conserve it immediately due to invariance of i and f

and therefore $\hat{T}$ is not unitary. This has significant physical consequences as the corresponding Lie Algebra for $\hat{T}$ is not hermitian any more. Hence the corresponding observables are not real any more. Sandwiching this equality between the states yields

$$\langle f|\hat{T}^\dagger\hat{T}|i\rangle = \langle f|i(\hat{T}^\dagger - \hat{T})|i\rangle$$
$$= i\langle f|\hat{T}^\dagger|i\rangle - i\langle f|\hat{T}|i\rangle = i\langle i|\hat{T}|f\rangle^* - i\langle f|\hat{T}|i\rangle$$
$$= i(2\pi)^4\delta^4(p_f - p_i)(M^*_{f\to i} - M_{i\to f}))$$

Consider the complete basis

$$\mathbb{1} = \sum_\alpha \int \mathrm{d}\Pi_\alpha \,|\alpha\rangle\langle\alpha| = \sum_\alpha \int \prod_{j\in\alpha} \frac{1}{2E_j}\frac{\mathrm{d}^3 p_j}{(2\pi)^3}\,|\alpha\rangle\langle\alpha|$$

of the Hilbert space, where we use a similar Π as above, and we sum over all (discrete) particle states and (continuous) momentum states. Inserting this unity into this, we receive

$$\langle f|\hat{T}^\dagger\hat{T}|i\rangle = \sum_\alpha \int \mathrm{d}\Pi_\alpha \langle f|\hat{T}^\dagger|\alpha\rangle\ \langle\alpha|\hat{T}|i\rangle$$
$$= i\sum_\alpha \int \mathrm{d}\Pi_\alpha (2\pi)^4\delta^4(p_f - p_a)(2\pi)^4\delta^4(p_i - p_a)M_{i\to\alpha}M^*_{f\to\alpha}$$

This gives us

$$(2\pi)^4\delta(p_i-p_f)(M_{i\to f}-M^*_{f\to i}) = i\sum_\alpha \int \mathrm{d}\Pi_\alpha (2\pi)^4\delta^4(p_f-p_a)(2\pi)^4\delta^4(p_i-p_a)M_{i\to\alpha}M^*_{f\to\alpha}$$

Therefore the generalised optical theorem is

$$M_{i\to f} - M^*_{f\to i} = i\sum_\alpha \int \mathrm{d}\Pi_\alpha (2\pi)^4\delta^4(p_f - p_a)M_{i\to\alpha}M^*_{f\to\alpha} \tag{2.15}$$

Now clearly the LHS is proportional to M, and is therefore also of order $O(g^n)$, where n is the highest number of the vertices considered in M. However, the RHS is proportional to $|M|^2$ and is therefore proportional to $O(g^{2n})$. For instance for ϕ^4 theory and considering $M \propto O(g^1)$, we have that the LHS is described by single vertex i.e. the tree-level, but the RHS is of order $O(g^2)$, so is already at one-loop level. Hence an important consequence of this general optical theorem, and even more generally, of unitary, is that certain loop expressions have to match tree level amplitudes [5].

Consider general *elastic* scattering, where by definition the initial and final particle and momentum states are the same because elastic scattering conserves the kinetic energy for each particle individually. For such n-n scattering, we have $|i\rangle = |f\rangle$, then equation above reduces to

$$2i\mathrm{Im}M_{i\to i} = M_{i\to i} - M^*_{i\to i} = i\sum_\alpha \int \mathrm{d}\Pi_\alpha (2\pi)^4\delta^4(p_f - p_a)|M_{i\to\alpha}|^2 \tag{2.16}$$

Recalling the optical theorem of first kind

$$\sigma_{i\to\alpha} = \frac{1}{4E_{CoM}p_i}\int \mathrm{d}\Pi_\alpha (2\pi)^4\delta(p_i - p_\alpha)|M_{i\to\alpha}|^2$$

we therefore derive the optical theorem of second kind

$$\mathrm{Im}M_{i\to i} = 2E_{CoM}p_i\Sigma_\alpha\sigma_{i\to\alpha} \tag{2.17}$$

The optical theorem of first kind relates $|M|^2 \sim \sigma$ and the generalised one relates $|M|^2 \sim \mathrm{Im}M$, so both combined imply the optical theorem of second kind, relating $\mathrm{Im}M \sim \sigma$. One can always derive one of the three from the other two. Now consider equation 2.17 again, here the RHS is clearly positive as $E_{\mathrm{CoM}} \geq 0$, hence the LHS must be too and we arrive at the positivity condition due to unitarity

$$\mathrm{Im}(M) > 0. \tag{2.18}$$

For $M = M(s)$ this means that one can only focus on the upper half plane as the amplitude cannot have a negative imaginary part for any s. However, one can of course analytically extend the amplitude to the lower half plane by using the Schwartz reflection principle [10]

$$A^*(s) = A(s^*), \tag{2.19}$$

which implies that flipping the amplitude along the Re(A) axis is equivalent to the initial amplitude at s^*.In summary

$$\begin{aligned} &\mathrm{Im}(M(s)) > 0 && \text{for upper half plane} \\ &\mathrm{Im}(M(s^*)) < 0 && \text{for lower half plane} \end{aligned} \tag{2.20}$$

Note that the optical theorem of second kind only holds for elastic scattering[6][5], whereas the other two hold for both elastic/inelastic scattering. Of course the one of second kind can be modified to include inelastic scattering, but then the equality becomes an inequality since there are now more scattering channels.

2.3.2 Analyticity

The second constraint that needs to be fulfilled by EFTs is analyticity. Generally analyticity refers to functions converging to their Taylor expansion. If the function is analytic everywhere, it is called an entire function. Of course there might be separate analytic regions that are topologically disconnected, that is, the convergence to the Taylor expansion exists inside these regions, but in between them the functions diverge, leading to singularities. Now clearly one can still consider all converging regions holistically by exploiting analytic extensions. However, the fact remains that there are still points or regions where divergencies do occur. Requiring our scattering amplitudes to be analytic is equivalent to requiring our amplitudes 1) the usual differentiable, converging to Taylor expansion and 2) ensuring we can always analytically extend our amplitudes across these divergencies, which makes sure that these singularities are dealt with. Hence requiring analyticity is necessary in order to consider a theory involving singularities. Alternatively, understanding the need for analyticity is also related to imposing theory to obey causality due to additional locality requirements to ensure that local operators are non-zero only inside light-cones, as explained in [11].

In order to derive bounds due to analyticity, it is useful to derive the Källén-Lehman representation, showing the regions where the amplitude is not analytic. Starting with the 2 point function $\langle\Omega|T\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle$, one can insert a complete set of states $\mathbb{1} = \Sigma_\alpha \int \frac{\mathrm{d}^4p}{(2\pi)^4} |\alpha, p\rangle \langle\alpha, p|$, labelled by discrete state α and momentum p, to get

$$\langle\Omega|T\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle = \Sigma_\alpha \int \frac{\mathrm{d}^4p}{(2\pi)^4} \langle\Omega|\hat{\phi}(x)|\alpha, p\rangle \ \langle\alpha, p|\hat{\phi}(y)|\Omega\rangle$$

Assuming Lorentz and translation invariance, i.e. a Poincare symmetry, it can be shown that

$$\langle\Omega|\hat{\phi}(x)|\Omega\rangle = e^{-ipx} \langle\Omega|\hat{\phi}(0)|\Omega\rangle$$

so therefore

$$\langle\Omega|T\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle = \Sigma_\alpha| \langle\alpha, p|\hat{\phi}(0)|\Omega\rangle |^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-i(x-y)}$$

$$\langle\Omega|T\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle = \int_0^\infty \mathrm{d}\frac{m^2}{2\pi} \rho(m^2) P_F(x, y)$$

This spectral representation shows the particle spectrum of a given theory. For 1 particle states there is one state with mass $m = m_\alpha$. This is where the Feynman propagator has a pole, and as a result, the spectral density has a Dirac delta spike at $s = m_\alpha$. One could argue this pole in the complex plane acts as the reason for particles becoming physical because at the pole, i.e. exactly at $s = p^2 = m_\alpha^2$ the particle obeys the on-shell condition. Particles going on-shell correspond to poles when their centre-of-mass energy s is equal to their particle masses. For 1 particle states this is very simple because the spectral density can only be a sum of Dirac functions, so visually a sum of Dirac spikes. Of course there can also be other bound states from composite particles, which would act like new 1 particle states giving additional Dirac spikes for $s > m^2$in the spectral representation, but the theories investigated in this dissertation are all made from elementary, i.e. no composite, particles. The minimum rest mass for 2 particle states clearly is $(2m)^2 = 4m^2$. Hence for $m^2 \leq s \leq 4m^2$ there are only 1 particle bound states, and for $s \geq 4m^2$ there are multiparticle states. These can be very complicated, and would go beyond the scope of this thesis, so from now on $s \leq 4m^2$ is assumed. As there are infinitely many combinations of particle masses in multiparticle states, that means that there are infinitely many poles along the real s axis, corresponding to a branch cut for $s \geq 4m^2$, so the amplitude cannot be analytic on the real axis for this region. Since we are focusing on 1 particle states at most, here this branch cut is less significant but still worth noting. Similarly for $t = 0$, crossing symmetry implies that there must be another pole at $u = m^2$ for the 1 particle state, and a branch cut for $u > 4m^2$. In terms of s these are $s = 3m^2$ and $s < 0$. To summarise, on the real axis, there are poles at $s = m^2$ and $s = 3m^2$ for 1 particle states and two branch cuts for multiparticle states, see Fig. 2.3. The poles and branch cuts are clearly not analytic, although the region between them is, and is called the unphysical region. However, it is very significant to see that these singularities only lie on the real axis, so the amplitude is still expected to be analytic *everywhere* else in the complex upper and lower half plane except for the physical regions on the real axis.

For the unphysical region $0 < s < 4m^2$ we know that M is analytic - except for $s = m^2$, giving the one-particle pole corresponding to the one-particle state going on-shell. Without this specific pole, M would be analytic everywhere for all s apart from the two branch cuts on the real axis, so we can now proceed by either assuming that this $s = m^2$ pole / 1 particle state didn't exist and then later correct the expression by adding in the pole [9] OR one treats M as not analytic at these two poles, which requires an additional deformation of the integration contour around these two and then immediately arrives at the corresponding residues. Clearly both methods should give the same result for the overall scattering amplitude. These poles/residues are due to particles going on-shell ($p^2 = m^2$) in the momentum Feynman propagators $\frac{i}{p^2-m^2+i\epsilon}$, so this is were these poles really come from. However, this also means that for any process that only involves the scattering of e.g. ϕ^4 theory to one-vertex-order, no such Feynman propagators appear in the amplitude as the external legs are simply given by *free* propagators only. Hence in such cases we do not have any poles at $s = m^2$ in the first place, and M is analytic everywhere anyway, apart from branch cuts. For such simple scattering, the difference between the actual amplitude and the pole-subtracted-amplitude is trivial. For all other cases we need to distinguish between M and S. To summarise, the positivity bounds derived above work for analytic regions only as the Cauchy integral theorem requires analytic regions, so any poles will either have to be subtracted before taking the derivatives, or directly taking care of by performing additional contour integrations, yielding additional residues that correspond to these subtractions. [10] Hence one can express

$$M(s_0) = \frac{1}{2\pi i} \oint_C \frac{M(s)}{s - s_0} ds \tag{2.21}$$

and for all s M is analytic apart from branch cuts.

[9] more specifically, the residue of the pole

[10] Also note that for multiparticle states the Dirac spikes are smeared out in the spectral representation because the particles can have arbitrary particle masses as long as $s > 4m^2$ and have more dofs, but when measuring a specific multiparticle state, their masses would immediately go back on-shell. Hence even though the spectral multiparticle shape appears to be finite and is, in fact, even related to cross sections, this does not imply the amplitude is analytic there. In some sense the spectral density shows which states are possible

Recalling Cauchy's integral formula

$$f(s_0) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s - s_0} ds \tag{2.22}$$

where $f(s)$ is analytic inside the curve C, as well as its nth derivative

$$f^{(n)}(s_0) = \frac{n!}{2\pi i} \oint_C \frac{f(s)}{(s - s_0)^{n+1}} ds, \tag{2.23}$$

Now it follows that we can directly see that for $n = 2$ we can rewrite the equation above as

$$M^{(2)}(s_0) = \frac{1}{\pi i} \oint_C \frac{M(s)}{(s - s_0)^3} ds, \tag{2.24}$$

or in a nicer form as

$$G(s_0) = \frac{M^{(2)}(s_0)}{2} = \frac{1}{2\pi i} \oint_C \frac{M(s)}{(s - s_0)^3} ds, \tag{2.25}$$

This now allows us to express the dispersion relation in terms of partial derivatives on the scattering amplitude. Clearly any (n) could have been chosen, but as shown below, $n = 2$ is the smallest n for which convergence is guaranteed, so this case corresponds to the bound of lowest order. In order to compute this integral explicitly, we let $s \to s + i\epsilon$ and $M(s) \to M(s(1 + i\epsilon))$, where $\lim_{\epsilon \to 0}$ is assumed. We can then use the contour in Fig. 2.2, where the epsilon ensures the curve is not on the poles and we arrive at

$$\begin{aligned} G(s_0) = \frac{1}{2\pi i} \Bigg(& \underbrace{\int_{-\infty}^{0} \frac{M(s(1+i\epsilon))}{(s(1+i\epsilon) - s_0)^3} \mathrm{d}s}_{G_1(s_0)} + \underbrace{\int_{0}^{-\infty} \frac{M(s(1-i\epsilon))}{(s(1-i\epsilon) - s_0)^3} \mathrm{d}s}_{G_2(s_0)} \\ & + \underbrace{\int_{4m^2}^{\infty} \frac{M(s(1+i\epsilon))}{(s(1+i\epsilon) - s_0)^3} \mathrm{d}s}_{G_3(s_0)} + \underbrace{\int_{\infty}^{4m^2} \frac{M(s(1-i\epsilon))}{(s(1-i\epsilon) - s_0)^3} \mathrm{d}s}_{G_4(s_0)} \\ & + \underbrace{\int_{C^+_{-\infty}} \frac{M(s(1+i\epsilon))}{(s(1+i\epsilon) - s_0)^3} \mathrm{d}s}_{G_5(s_0)} + \underbrace{\int_{C^-_{\infty}} \frac{M(s(1-i\epsilon))}{(s(1-i\epsilon) - s_0)^3} \mathrm{d}s}_{G_6(s_0)} \Bigg) \end{aligned} \tag{2.26}$$

Again, this is based on the assumption that we can decompose the contour integral $\oint_C$ as such a sum of ordinary integrals, which works as long as $M(S)$ is analytic inside this contour, i.e. when neglecting the $s = m^2$ pole, and then adding the residue later on. If we included the single particle creation poles in the first place, we would have to deform the contour a bit more, so we would have more terms to consider in the sum of integrals. It is these additional integrals that give us the expression for the residue we use to add in the former case, so both approaches should give the same result.

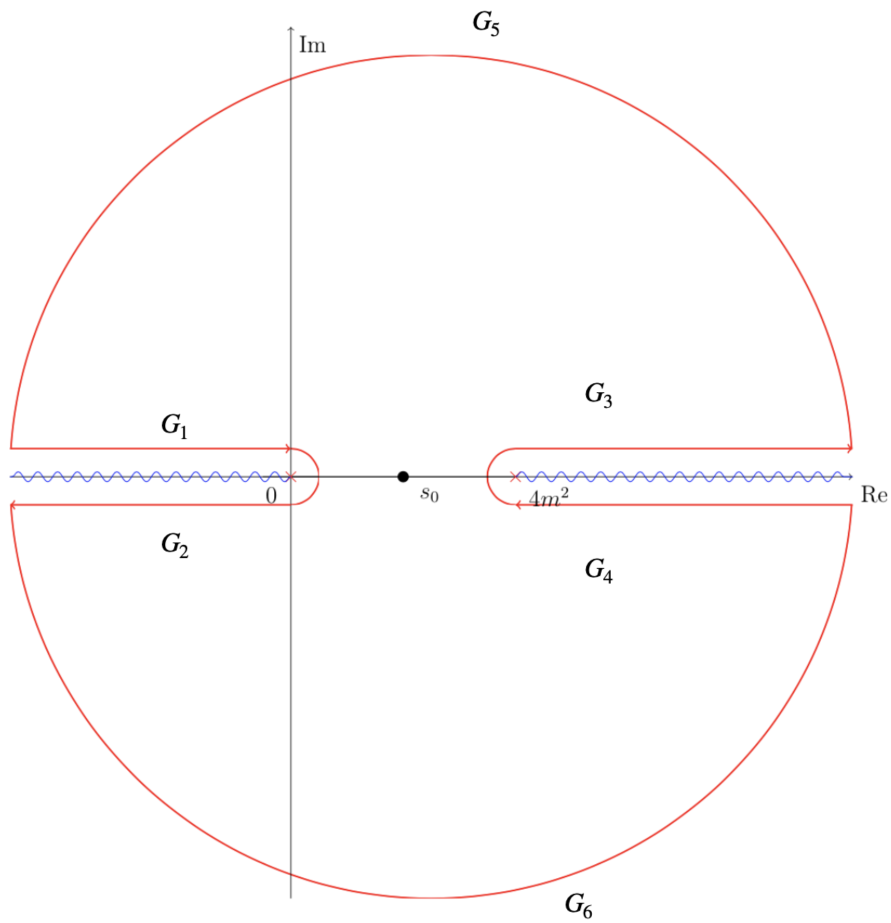

Figure 2.2: Reproduced and edited from [10], this sketch shows the integration contour and also shows the two branch cuts for $s < 0$ and $s > 4m^2$.

The Froissart bound [12] [13] implies that $M(s_0)$ is bounded from above by $|M(s)| < sln^2(s)$ in the limit $lim_{s\to\infty}$. Now integrating along the infinitely big circle C, the integral measure is proportional to $s ds$, so $lim_{s\to\infty}|\frac{M}{(s-s_0)^3}ds| = lim_{s\to\infty}\frac{ln^2(s)}{s} = 0$ Hence this directly implies that $G_4(s_0) \to 0$ and $G_5(s_0) \to 0$, i.e. the boundary terms vanish. At this point it is important to reconsider equation 2.23 and note that if one had chosen $n = 1$, ie poles of second order in $M(s_0) = \frac{1}{2\pi i}\oint_C \frac{M(s)}{(s-s_0)^2}ds$ only, then we would get boundary terms proportional to $lim_{s\to\infty}ln^2(s) = \infty$ i.e. not just boundary terms that did not vanish, but also boundary terms that would entirely diverge! Hence $n = 2$ is the smallest order state where the integral converges, and in general one could consider any $n \geq 2$ that will then give additional, higher-order bounds as explained below.

In summary one can write

$$G(s_0) = \frac{1}{2\pi i}\left(\int_{-\infty}^{0} \frac{M\big(s(1+i\epsilon)\big) - M\big(s(1-i\epsilon)\big)}{(s(1+i\epsilon)-s_0)^3}\mathrm{d}s + \int_{4m^2}^{\infty} \frac{M\big(s(1+i\epsilon)\big) - M\big(s(1-i\epsilon)\big)}{(s(1-i\epsilon)-s_0)^3}\mathrm{d}s\right) \tag{2.27}$$

$$= \frac{1}{2\pi i}\left(\int_{-\infty}^{0} \frac{Disc(M(s))}{(s(1+i\epsilon)-s_0)^3}\mathrm{d}s + \int_{4m^2}^{\infty} \frac{Disc(M(s))}{(s(1-i\epsilon)-s_0)^3}\mathrm{d}s\right) \tag{2.28}$$

where the discontinuity function is defined as $Disc(M) := M\big(s(1+i\epsilon)\big) - M\big(s(1-i\epsilon)\big)$. After simplifying the numerators this way, the next step is to change the integration limits so that both terms have the same limit. Hence we conduct change of variables in the first integral with $u = 4m^2 - s$, so $s = \infty \Rightarrow u = \infty$ and $s = 0 \Rightarrow u = 4m^2$ and $\mathrm{d}u = \mathrm{d}s$. This then gives

$$= \frac{1}{2\pi i}\int_{4m^2}^{\infty} \frac{-Disc(M(4m^2-u))}{((4m^2-u)(1+i\epsilon)-s_0)^3}\mathrm{d}u + \int_{4m^2}^{\infty} \frac{Disc(M(s))}{(s(1-i\epsilon)-s_0)^3}\mathrm{d}s \tag{2.29}$$

$$= \frac{1}{2\pi i}\int_{4m^2}^{\infty} \left(\frac{Disc(M(s))}{(s(1-i\epsilon)-s_0)^3}\mathrm{d}s - \frac{Disc(M(4m^2-u))}{((4m^2-u)(1+i\epsilon)-s_0)^3}\mathrm{d}u\right) \tag{2.30}$$

Now we exploit crossing symmetry, which requires that $M(s) = M(u)$, so $Disc(M(s))) = Disc(M(u))$ to get

$$= \frac{1}{2\pi i}\int_{4m^2}^{\infty} \left(\frac{Disc(M(s))}{(s(1-i\epsilon)-s_0)^3}\mathrm{d}s - \frac{Disc(M(s))}{((4m^2-u)(1+i\epsilon)-s_0)^3}\mathrm{d}u\right) \tag{2.31}$$

and note that the first term depends on s only, and the second term on u only. Thus we can now relabel $u \to s$ in the second term. Then this integral becomes

$$= \frac{1}{2\pi i}\int_{4m^2}^{\infty} Disc(M(s))\left(\frac{1}{(s(1-i\epsilon)-s_0)^3} - \frac{1}{((4m^2-s)(1+i\epsilon)-s_0)^3}\right)\mathrm{d}s \tag{2.32}$$

It is clear that

$$2i\mathrm{Im}(M(s(1+i\epsilon)) = M(s(1+i\epsilon)) - M^*(s(1+i\epsilon)) = M(s(1+i\epsilon)) - M(s(1-i\epsilon)) = Disc(M(s)) \tag{2.33}$$

so one can write

$$G(s_0) = \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{Im}(M(s))\left(\frac{1}{(s(1-i\epsilon)-s_0)^3} - \frac{1}{((4m^2-s)(1+i\epsilon)-s_0)^3}\right)\mathrm{d}s \quad (2.34)$$

Now using the optical theorem of second kind $\mathrm{Im}(M(s)) = \mathrm{Im}(M(i \to i)) = 2E_{cm}|\vec{p}_i|\Sigma_\alpha\sigma(\alpha \to i)$

$$G(s_0) = \frac{2}{\pi}\int_{4m^2}^{\infty} E_{cm}|\vec{p}_i|\Sigma_\alpha\sigma(\alpha \to i)\left(\frac{1}{(s(1-i\epsilon)-s_0)^3} - \frac{1}{((4m^2-s)(1+i\epsilon)-s_0)^3}\right)\mathrm{d}s \quad (2.35)$$

Here very clearly$E_{cm} \geq 0$, $|\vec{p}_i| \geq 0$ and $\sigma(\alpha \to i) \geq 0$. Considering the bracket, in the exact limit $\epsilon = 0$ the bracket is just

$$\frac{1}{(s-s_0)^3} - \frac{1}{(4m^2-s-s_0)^3} = \frac{1}{(s-s_0)^3} + \frac{1}{(s+s_0-4m^2)^3} \geq 0$$

as the integration limits in eqn imply that integration is between $4m^2$ and ∞, so here $s \geq 4m^2$, and the second term in the bracket is also positive. Putting all together we get

$$G(s_0) = \frac{2}{\pi}\int_{4m^2}^{\infty} E_{cm}|\vec{p}_i|\Sigma_\alpha\sigma(\alpha \to i)\left(\frac{1}{(s(1-i\epsilon)-s_0)^3} - \frac{1}{((4m^2-s)(1+i\epsilon)-s_0)^3}\right)\mathrm{d}s \geq 0 \quad (2.36)$$

Assuming we ignore the obviously trivial cases like $E_{cm} = 0$, $p = 0$ or $\sigma = 0$ ie assuming the scattering process always happens, we get the even stronger bound

$$G(s_0) = \frac{2}{\pi}\int_{4m^2}^{\infty} E_{cm}|\vec{p}_i|\Sigma_\alpha\sigma(\alpha \to i)\left(\frac{1}{(s(1-i\epsilon)-s_0)^3} - \frac{1}{((4m^2-s)(1+i\epsilon)-s_0)^3}\right)\mathrm{d}s > 0 \quad (2.37)$$

Now recall $G(s_0) = \frac{M^{(2)}(s_0)}{2} = \frac{\partial^2 M(s_0)}{\partial s_0^2}$ from eqn xx Relabelling $s_0 \to s$ and given that $G(s_0) > 0$ we have

$$\frac{\partial^2 M(s)}{\partial s^2} > 0 \quad (2.38)$$

This the positivity condition, significantly constraining the coupling conditions in the theory. For any $n \geq 2$, it was additionally shown above that convergence of the integrals is guaranteed, and using a similar argument one derive higher-order positivity bounds of the form

$$\frac{\partial^n M(s)}{\partial s^n} > 0. \quad (2.39)$$

It is immediately apparent that this results in an infinite number of bounds, however, given any explicit amplitude $A(s)$ of order $O(s^m)$ one gets exactly $|m-1|$ different, non-trivial bounds[11].

[11] One could argue that this still corresponds to infinitely many bounds, but just $|m-1|$ non-trivial ones and infinitely many trivial ones of the form $0 \geq 0$

As discussed before, this only works if $M(s)$ is analytic for all s. For any theory to make sense, it must always include the 1 particle state, so there must always be a pole at $s = m^2$. If one adds in this pole, then the contour would have to be deformed like in Fig. 2.3.

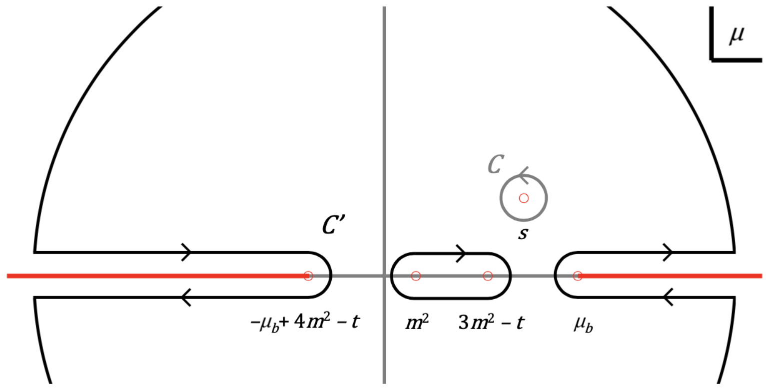

Figure 2.3: Reproduced from [6], this sketch visualises the different integration contour with poles at $s = m^2$ and $s = 3m^2$ for $t = 0$. μ_b can be chosen to be $\mu_b = 4m^2$ so that this point coincides with the start of the multiparticle branch cuts.

In chapter 3 it will be shown how to calculate the corresponding residues for these poles.

Chapter 3

Scalar bounds

3.1 Simple example

It is best to start applying positivity bounds to scalar theories before dealing with more complicated vector theories later, so consider a general ϕ^4 theory with $H^I_{int} = \hat{O}(\partial^4, \phi^4)$, i.e. quartic derivative interactions. There are many ways of obtaining combinations of derivative couplings with dimension 8, such as $H^I_{int} = \lambda \partial_\mu \phi \partial^\nu \partial^\mu \phi \partial_\nu \phi \phi$ or

$$H^I_{int} = \lambda \partial_\mu \phi \partial^\mu \phi \partial_\nu \phi \partial^\nu \phi \tag{3.1}$$

It is crucial to distinguish the $s-$, $t-$ and $u-$channels at this point and label the particles carefully. Unlike for cubic interactions, where the definition of the channels is obvious, for quartic vertices assigning the channels is less clear and has a certain degree of ambiguity. However, cubic theories have extra complications due to poles at tree-level order, so in this example only quartic interactions are studied. For the s-channel one needs at least one factor of the Mandelstam variable s, i.e. at least one contraction $p_1 \cdot p_2$ or $p_3 \cdot p_4$. Therefore it is sensible to label the momenta of the terms with the μ contractions with 1 and 2 and the ones with the ν contractions with 3 and 4:

$$H^I_{int} = \lambda \underbrace{\partial_\mu \phi}_{p_1} \underbrace{\partial^\mu \phi}_{p_2} \underbrace{\partial_\nu \phi}_{p_3} \underbrace{\partial^\nu \phi}_{p_4} \tag{3.2}$$

The Feynman diagram for this interaction is . Given that there are two incoming particles (1 and 2) that have negative momenta $(-ip_1)$ and $(-ip_2)$ and two outgoing particles (3 and 4) with positive momenta $(+ip_3)$ and $(+ip_4)$, their overall product is positive as the signs cancel. Each channel has 8 permutations as there are $4! = 24$ permutations in total and 3 channels. The scattering amplitude according to the Feynman rules then is

$$M = 8\lambda \Big((-ip_1) \cdot (-ip_2) \Big) \Big(ip_3 \cdot ip_4 \Big), \tag{3.3}$$

where again terms like $(p_i \cdot p_j)$ mean $(p_i \cdot p_j) = p_{i,\mu} p_j^\mu$ Now invoking equation 3.3 this can be rewritten as

$$M^s(s) = 8\lambda(p_1 \cdot p_2)(p_3 \cdot p_4) = 8\lambda\Big(\frac{s}{2} - m^2\Big)^2 = 8\lambda\Big(\frac{s^2}{4} - m^2 s + m^4\Big) \tag{3.4}$$

For the t-channel one obtains a similar expression, however, here one needs at least one factor of t, i.e. a $p_1 \cdot p_3$ or $p_2 \cdot p_4$ contraction. Hence one relabel the terms so that p_1 and p_3 share the μ contractions and p_2 and p_4 share the ν contractions, and then derive everything again from this, or directly use the result for the s-channel and there commit to the relabelling $2 \leftrightarrow 3$. Either method gives

$$M^t(t) = 8\lambda(p_1 \cdot p_3)(p_2 \cdot p_4) = 8\lambda\Big(m^2 - \frac{t}{2}\Big)^2 = 8\lambda\Big(\frac{t^2}{4} - m^2 t + m^4\Big) \tag{3.5}$$

Similarly for the u-channel, one either relabels $3 \to 4$ *in the s-channel result*, or relabels $2 \to 4$ *in the $t-$channel result*. Either then results in

$$M^u(u) = 8\lambda(p_1 \cdot p_4)(p_2 \cdot p_3) = 8\lambda\Big(m^2 - \frac{u}{2}\Big)^2 = 8\lambda\Big(\frac{u^2}{4} - m^2 u + m^4\Big) \tag{3.6}$$

Now setting $t = 0$ and using $u = 4m^2 - s$ one can express all amplitudes as functions of s only, giving

$$\begin{aligned} M^s(s) &= 8\lambda\Big(\frac{s^2}{4} - m^2 s + m^4\Big) \\ M^t(s) &= 8\lambda m^4 \\ M^u(s) &= 8\lambda\Big(\frac{s^2}{4} - 3m^2 s + m^4\Big) \end{aligned} \tag{3.7}$$

Note that there are no poles in the amplitudes, as expected, so the pole-subtracted amplitude S is just the ordinary amplitude $M = S$. However, if one wishes to add in a cubic interaction at two-vertex order, then pole-subtractions are necessary to remove the infinities.

Hence the lowest order positivity bounds are

$$\frac{\partial^2 S}{\partial s^2} = \frac{\partial^2 M}{\partial s^2} = \frac{\partial^2 (M^s + M^t + M^u)}{\partial s^2} = 16\lambda > 0 \tag{3.8}$$

and hence

$$\lambda > 0 \tag{3.9}$$

This is a new constraint on the coupling constant λ, that was unspecified before. One could now consider some higher-dimensional Lagrangians with a higher number of derivatives acting on the four ϕ fields, giving more momentum contractions and hence higher polynomials in s. Generally, higher order bounds $\frac{\partial^n S}{\partial s^n} > 0$ can then be additionally applied for scalar terms of dimension $2n+4$ with $n \geq 2$ in the Lagrangian. This is because given that both scalars and single derivatives have mass dimension $+1$,

n pairs of contracted derivatives acting on 4 scalar fields then give dimension $2n+4$. Only the $2n$ derivatives give momentum factors in the amplitude, so a Lagrangian density of dimension $2n+4$ gives an amplitude of order $O(p^{2n}) = O(s^n)$. Therefore non-trivial, higher order bounds up to $\frac{\partial^n S}{\partial s^n} > 0$ are possible.

Instead of using higher-order bounds, one can also focus on the lowest order bound $\frac{\partial^2 S}{\partial s^2} > 0$, which will be bounds in terms of s. This is not the case for the higher order ones as the additional derivatives remove the s-dependency entirely.

3.2 Cubic and quartic Lagrangian with higher dimension

Consider the Lagrangian with the following cubic and quartic interactions:

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + g_3^0(\partial\phi)^2\Box\phi - g_4^0(\partial\phi)^2((\Box\phi)^2 - (\partial_{\mu\nu})^2) \tag{3.10}$$

The fact that the action $S = \int \mathrm{d}^4 x L$ is dimensionless allows to infer from the mass and kinetic terms that both ϕ and ϕ_μ have mass dimension 1 in natural units, i.e are of order $O(m^1)$. Hence it is apparent that the mass and kinetic terms are dimension 4, so $g_3^0(\partial\phi)^2\Box\phi$ has to be of order $O(m^4)$ as well. This implies that g_3^0 is $O(m^3)$, so one can then factor out the dimensionless part of g_3^0 by dividing by the mass scale Λ cubed. This motivates to explicitly write $g_3 = \frac{g_3^0}{\Lambda^3}$. Similarly, this means that for the quartic term g_4^0 is of order $O(m^6)$. Therefore the dimensionless quartic coupling is $g_4 = \frac{g_4^0}{\Lambda^6}$

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g_3}{\Lambda^3}(\partial\phi)^2\Box\phi - \frac{g_4}{\Lambda^6}(\partial\phi)^2((\Box\phi)^2 - (\partial_{\mu\nu})^2) \tag{3.11}$$

Whilst equation 3.10 and 3.11 are equivalent, if these Lagrangians represent an EFT, then only the last one makes the scale dependence obvious in this form.

It also important to take into account cases with different particles masses, where each particle i will have its own mass m_i and associated energy scale Λ_i. Then factoring out mass scales to make the coupling constant dimensionless allows to immediately show which particle masses become relevant at specific energies and what the particle spectrum below a given Λ_i is.

From the considerations in chapter 2.2 one expects that in the UV limit above some cut-off Λ any quartic term would actually be described by cubic terms[1]. Hence this would directly allow to assume that g_3 and g_4 are not independent of each other. In fact, as will be shown below, the cubic term can actually be *absorbed* into the quartic term via a field redefinition.

[1]This was already considered for a similar case of quartic vertices that became sextic vertices, however, the analogy is clear.

3.2.1 Positivity bound

In order to find the corresponding positivity bounds, an expression for the amplitude as a function of s has to be found first. The Feynman rules for derivative couplings as explained above still apply. Specifically, for terms like $L \supset g\Box\phi = g\partial_\mu\partial^\mu\phi$ the contribution to the scattering amplitude is $\pm i^2 g p_\mu p^\mu = \mp g p^2 = \mp g m^2$, depending on whether it is incoming or outgoing.

Firstly, consider the quartic term,

$$L_4 = -\frac{g_4}{\Lambda^6}(\underbrace{(\partial\phi)^2(\Box\phi)^2}_{L_{4a}} - \underbrace{(\partial\phi)^2(\partial_\mu\partial_\nu\phi)^2}_{L_{4b}}, \tag{3.12}$$

for L_{4a} one has for the s-channel

$$M^s_{L_{4a}} = -8i\frac{g_4}{\Lambda^6}\big((-ip_1)\cdot(-ip_2)\big)\big(ip_3\big)^2\big(ip_4\big)^2 = 8i\frac{g_4}{\Lambda^6}m^4(p_1\cdot p_2) = 4i\frac{g_4}{\Lambda^6}m^4(s-2m^2)$$

Similarly, swapping $2 \leftrightarrow 3$ in the s channel gives

$$M^t_{L_{4a}} = -8i\frac{g_4}{\Lambda^6}\big((-ip_1)\cdot(-ip_3)\big)\big(ip_2\big)^2\big(ip_4\big)^2 = 8i\frac{g_4}{\Lambda^6}m^4(p_1\cdot p_3) = 4i\frac{g_4}{\Lambda^6}m^4(2m^2-t)$$

and then swapping $3 \leftrightarrow 4$ results in the u-channel:

$$M^u_{L_{4a}} = -8i\frac{g_4}{\Lambda^6}\big((-ip_1)\cdot(-ip_4)\big)\big(ip_2\big)^2\big(ip_3\big)^2 = 8i\frac{g_4}{\Lambda^6}m^4(p_1\cdot p_4) = 4i\frac{g_4}{\Lambda^6}m^4(2m^2-u)$$

Note that, e.g by going from the s-channel to the t channel the amplitudes do not pick up any other signs. This is because in the s channel, 1 and 2 are incoming and have allocated a single derivative acting on the fields, so p_1 and p_2 pick up a minus sign each, and 3 and 4 have a box each. This is whilst in the t channel, now particles 1 and 3 are incoming and get directly contracted, so now particle 2 gets a box applied on its field. Considerations like this are significant and easily overlooked, but it is very easy to omit signs, so it is always best to write down the amplitude in terms of all fields *and then simplify it* for each channel, opposed to just evaluating the s channel and then invoking crossing symmetry and relabelling. If, however, there were only single derivatives acting on fields in the Lagrangian (such as for cases discussed in chapter 4), then it is a lot easier to see that the overall sign factor for 2-2 scattering with quartic vertices is always[2] going to be $(+i)(+i)(-i)(-i) = +1$ as there won't be any boxes acting on ϕ giving squares of some permutations any more.

For the second term L_{4b}one obtains by using these methods

[2]One could argue that for vertices with an odd number of external legs the additional i next to the coupling constant would be relevant and might give the opposite sign. However, whilst e.g. quintic vertices are possible, for one-vertex diagrams the number of derivatives has to be even because every derivative contraction requires 2 indices, which is even, so the expression above should generally hold.

$$M^s_{4_b} = 8i\frac{g_4}{\Lambda^6}\big((-ip_1)\cdot(-ip_2)\big)\big(ip_{3,\mu}ip_{3,\nu}\big)\big(ip_4^\mu ip_4^\nu\big)$$
$$= -8i\frac{g_4}{\Lambda^6}(p_1\cdot p_2)(p_3\cdot p_4)^2 = -i\frac{g_4}{\Lambda^6}(s-2m^2)^3$$

and for the t and u channels

$$M^t_{4_b} = -8i\frac{g_4}{\Lambda^6}(p_1\cdot p_3)(p_2\cdot p_4)^2 = -i\frac{g_4}{\Lambda^6}(2m^2-t)^3$$

and

$$M^u_{4_b} = -8i\frac{g_4}{\Lambda^6}(p_1\cdot p_4)(p_2\cdot p_3)^2 = -i\frac{g_4}{\Lambda^6}(2m^2-u)^3$$

Now consider the cubic term

$$L_3 = \frac{g_3}{\Lambda^3}(\partial\phi)^2\Box\phi \tag{3.13}$$

Now the tree-level diagram for 2-2 scattering has two vertices, so one needs to calculate the scattering cross section for the Feynman diagram given in Fig. xx. In the s - channel, 1 and 2 contract as always, however, there is now the further complication of undetermined momenta in the propagator corresponding to virtual particles. As explained before, these give rise to poles. Specifically, the amplitude is

$$L^s_3 = 8\frac{(ig_3)^2}{\Lambda^6}\left((-ip_1)\cdot(-ip_2)(i(p_1+p_2))^2)\right)\left(\frac{i}{(p_1+p_2)^2-m^2+i\epsilon}\right)\left((-i(p_3+p_4))^2(ip_3\cdot ip_4\right) \tag{3.14}$$

as due to conservation of momentum, $q^\mu = p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$ Recalling that $s = (p_1+p_2)$ and $p_1\cdot p_2 = p_3\cdot p_4 = \frac{s}{2} - m^2$ this gives

$$\begin{aligned} M^s_3 = \frac{-8ig_3^2}{\Lambda^6}(p_1\cdot p_2)(p_3\cdot p_4)\frac{s^2}{s-m^2+i\epsilon} &= -8i\frac{g_3^2}{\Lambda^6}\frac{s^2}{s-m^2+i\epsilon}(\frac{s-2m^2}{2})^2 \\ &= -8i\frac{g_3^2}{\Lambda^6}\frac{s^2}{s-m^2+i\epsilon}(\frac{s^2}{4}-sm^2+m^4) \end{aligned} \tag{3.15}$$

For the t channel swapping the indices gives a similar result, where now conservation of momentum implies $q^\mu = p_1^\mu - p_3^\mu = p_4^\mu - p_2^\mu$ for the virtual particle's momentum.

$$\begin{aligned} M^t_3 = \frac{-8ig_3^2}{\Lambda^6}(p_1\cdot p_3)(p_2\cdot p_4)\frac{t^2}{t-m^2+i\epsilon} &= -8i\frac{g_3^2}{\Lambda^6}\frac{t^2}{t-m^2+i\epsilon}(\frac{2m^2-t}{2})^2 \\ &= -8i\frac{g_3^2}{\Lambda^6}\frac{t^2}{t-m^2+i\epsilon}(\frac{t^2}{4}-tm^2+m^4) \end{aligned} \tag{3.16}$$

and finally

$$\begin{aligned} M^u_3 = \frac{-8ig_3^2}{\Lambda^6}(p_1\cdot p_4)(p_2\cdot p_3)\frac{u^2}{u-m^2+i\epsilon} &= -8i\frac{g_3^2}{\Lambda^6}\frac{u^2}{u-m^2+i\epsilon}(\frac{2m^2-u}{2})^2 \\ &= -8i\frac{g_3^2}{\Lambda^6}\frac{u^2}{u-m^2+i\epsilon}(\frac{u^2}{4}-um^2+m^4) \end{aligned} \tag{3.17}$$

Summing up all amplitudes

$$M = \frac{-8ig_3^2}{\Lambda^6}\Bigg(\frac{u^2}{u-m^2+i\epsilon}(\frac{u^2}{4} - um^2 + m^4) + \frac{t^2}{t-m^2+i\epsilon}(\frac{t^2}{4} - tm^2 + m^4)$$
$$+\frac{s^2}{s-m^2+i\epsilon}(\frac{s^2}{4} - sm^2 + m^4\Bigg) - \frac{ig_4}{\Lambda^6}\Bigg(- 4m^4(s-2m^2) - 4m^4(2m^2-t) \tag{3.18}$$
$$-4m^4(2m^2-u) + (s-2m^2)^3 + (2m^2-u)^3 + (2m^2-t)^3\Bigg)$$

Invoking the forward limit $t = 0$ as well as $u = 4m^2 - s$ and only keeping terms of at least $O(s^2)$, then this reduces to

$$M = \frac{-8ig_3^2}{\Lambda^6}\Bigg(\frac{s^2-8m^2s+16m^4}{3m^2-s+i\epsilon}(\frac{s^2}{4} - sm^2 + m^4)$$
$$+\frac{s^2}{s-m^2+i\epsilon}(\frac{s^2}{4} - sm^2 + m^4) + O(s^1)\Bigg) + \frac{2ig_4}{\Lambda^6}((s-2m^2)^3 + O(s^1)) \tag{3.19}$$

This very clearly has poles at $s = m^2$ and $s = 3m^3$. As described above, this is because so far in the analyticity calculation, M was assumed to be analytic everywhere on the real axis, so these 1 particle states were neglected. Now these poles have to be subtracted from this amplitude in order to rectify this step, and ensure the positivity bounds are applicable to this amplitude.

In general, the structure of such pole subtraction is [14]

$$S = M - \frac{\lambda}{m^2-s} - \frac{\lambda}{m^2-u} - \frac{\lambda}{m^2-t} \tag{3.20}$$

with the pole subtracted amplitude S, the full amplitude M and the residues λ . Ignoring the third term for forward $t = 0$, one would expect these residues to be related via crossing symmetry

$$\lambda = \mathrm{Res}_{u=m^2} = -\mathrm{Res}_{s=m^2} \tag{3.21}$$

but it always is best to explicitly calculate each residue directly to verify this equality.

Note that λ has the opposite sign for each residue because the denominator read as $m^2 - s$ and $m^2 - u = s - 3m^2$ so the s have opposite sign in the denominator, which will shortly be important.

Suppose $f(s) = \frac{g(s)}{h(s)}$ is a function of s with $g(s)$ and $h(s)$ analytic at s_0 and $h(s_0) = 0$, so that $f(s)$ has a simple pole at s_0. Then residues at $s = s_0$ are calculated [16]with

$$\mathrm{Res}_{s=s_0} = \lim_{s\to s_0}(s-s_0)f(s) = \lim_{s\to s_0}\frac{sg(s) - s_0g(s)}{h(s)}$$

$$= \lim_{s\to s_0}\frac{g(s)+sg'(s)-s_0g'(s)}{h'(s)} = \lim_{s\to s_0}\frac{g(s)+g'(s)(s-s_0)}{h'(s)} = \frac{g(s_0)}{h'(s_0)}$$

with L'Hospital rule. Hence the algorithm to calculate such a residue is to take the denominator, plug in the value of s where there is a pole, and divide it by the derivative of the denominator at s_0. For the cases above, $h_1(s) = m^2 - s$, so $h_1'(m^2) = -1$ and $h_2(s) = s-3m^2$, so $h_1'(3m^2) = 1$. This now allows to evaluate the poles in the following way

$$\begin{aligned}
\mathrm{Res}_{s=m^2} &= \mathrm{Res}_{s=m^2}\frac{s^2}{s-m^2}\Big(\frac{s^2}{4} - sm^2 + m^4\Big) \\
&= (-1)(s^2)\Big(\frac{s^2}{4} - sm^2 + m^4\Big)|_{s=m^2} = \frac{m^8}{4} \\
\mathrm{Res}_{s=3m^2} &= \mathrm{Res}_{s=3m^2}\frac{s^2-8m^2s+16m^4}{3m^2-s}\Big(\frac{s^2}{4} - sm^2 + m^4\Big) \\
&= \Big(s^2-8m^2s+16m^4\Big)\Big(\frac{s^2}{4} - sm^2 + m^4\Big)|_{s=3m^2} = -\frac{m^8}{4}
\end{aligned} \tag{3.22}$$

This shows that indeed $\lambda = \mathrm{Res}_{u=m^2} = -\mathrm{Res}_{s=m^2}$. Now that the poles are removed, in the following one can directly set all remaining *epsilon* in the propagators to $\epsilon = 0$, instead of just taking the limit $\epsilon \to 0$. The pole-subtracted amplitudes due to the cubic term are then

$$\begin{aligned}
S = \frac{-8ig_3^2}{\Lambda^6}\Bigg(&\frac{s^2-8m^2s+16m^4}{3m^2-s}(\frac{s^2}{4} - sm^2 + m^4) + \frac{m^8}{4}\frac{1}{s-3m^2} \\
&+\frac{s^2}{s-m^2}(\frac{s^2}{4} - sm^2 + m^4) - \frac{m^8}{4}\frac{1}{s-m^2}\Bigg) - \frac{2ig_4}{\Lambda^6}((s-2m^2)^3 + O(s)
\end{aligned} \tag{3.23}$$

These are finite for any $s \to \pm\infty$ now and plotting the behaviour of the amplitudes confirms the finite and converging nature of these subtracted amplitudes S, see Fig. 3.1 and Fig. ??.

Now that analyticity is guaranteed[3], the positivity bound is applicable, so calculating the second derivatives yields

$$-\frac{8ig_3^2}{\Lambda^6}\Big(-\frac{6s-18m^2}{4} + \frac{3s-3m^2}{2}\Big) - \frac{12ig_4}{\Lambda^6}(s-2m^2) > 0$$

These bounds are valid for the entire region where the amplitude is analytic, so must also hold for $s = 0$. Then this bound simply implies

$$g_3^2 - g_4 < 0$$

This is still for the $(+---)$ metric.

[3]except for any 2 particle states, and we chose to omit those for the scope of this project.

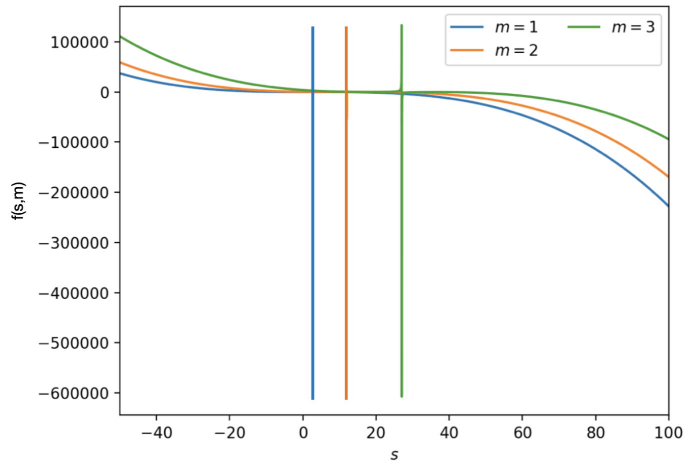

Figure 3.1: This plot shows the behaviour of the amplitude $f(s,m) = M(s) = \frac{s^2-8m^2s+16m^4}{3m^2-s}(\frac{s^2}{4} - sm^2 + m^4)$ with the pole at $s = 3m^2$ for several different masses. Clearly for each m there is a corresponding pole and the amplitude is not analytic everywhere.

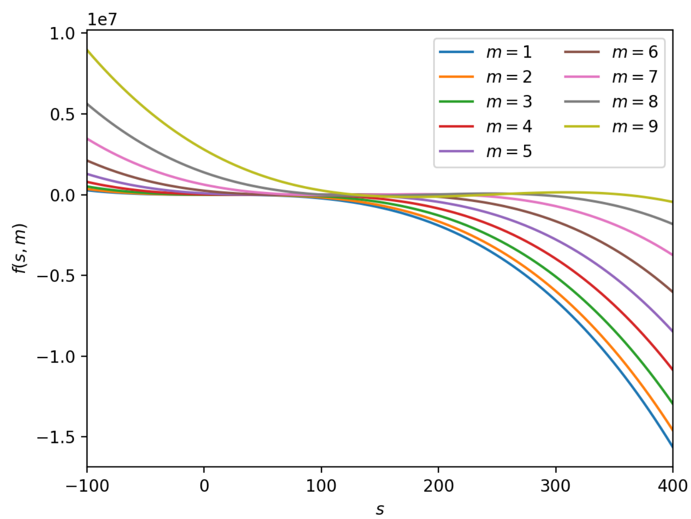

Figure 3.2: This plot shows the amplitude after the pole subtraction $f(s,m) = \frac{s^2-8m^2s+16m^4}{3m^2-s}(\frac{s^2}{4}-sm^2+m^4)+\frac{m^8}{4}\frac{1}{s-3m^2}$. There are no poles any more and the amplitudes are polynomial functions that are now finite and analytic at $s = 3m^2$.

If the other metric $(-+++)$ was used, then in the Lagrangian the kinetic and quartic terms have one term with an odd number of derivative contractions (the $(\partial\phi)^2$ terms[4]), so both would pick up a minus sign each. The cubic term has 2 index contractions, picking up $(-1)^2 = 1$, i.e. remains unchanged. For $(-+++)$ the results are then $g_3^2 + g_4 < 0$.

This is the positivity constraint corresponding to the lowest-order bound $\frac{\partial^2 S}{\partial s^2} > 0$.

3.2.2 Field redefinition

Field redefinition invariance is a well-known symmetry physical theories should obey. Consider making the field redefinition

$$\phi \to \phi' = \phi + \frac{g_3}{\Lambda^3}(\partial\phi)^2 + \frac{2g_3^2}{\Lambda^6}\partial_\alpha\phi\partial_\beta\phi\partial^\alpha\partial^\beta\phi \tag{3.24}$$

In the following I will then reproduce the superposition of coupling constants in the positivity bound above by absorbing the cubic term into the quartic one with this redefinition.

For this it is important to recall that for the lowest order positivity bound $s = 0$ was used. Hence the energy of the system was much smaller than any particle masses, $s \leq m^2$. However, the field redefinition only works for $m = 0$, which is fine because the positivity bounds should hold[5] for *any* s and *any* m. Hence the mass term in the Lagrangian can be ignored for the purpose of this field redefinition. By doing so one clearly gives up some generality, in that the redefinition is only applicable for massless particles, but this loss of generality gets traded in for an additional confirmation of the coefficients in the bounds in the massless case.

Hence the Lagrangian is then taken to be

$$L = -\frac{1}{2}(\partial\phi)^2 + g_3^0(\partial\phi)^2\Box\phi - g_4^0(\partial\phi)^2((\Box\phi)^2 - (\partial_{\mu\nu})^2) \tag{3.25}$$

In the following we expand in ϕ and then neglect all smaller terms of at least order $O(\phi^5)$ since such terms cannot be absorbed into the quartic one any more, which is already of $O(\phi^4)$.

Then the redefinition implies

$$(\partial_\mu\phi)(\partial^\mu\phi) \to (\partial_\mu\phi)(\partial^\mu\phi) + \frac{g_3^2}{\Lambda^6}\left[\partial_\mu(\partial\phi)^2\right]\left[\partial^\mu(\partial\phi)^2\right] + 2\frac{g_3}{\Lambda}\partial_\mu\phi\partial^\mu(\partial\phi)^2 + \frac{2g_3^2}{\Lambda^6}(\partial_\mu\phi\partial^\mu\left[\partial_\alpha\phi\partial_\beta\phi\partial^\alpha\partial^\beta\phi\right]) \tag{3.26}$$

[4]the quadratic term technically has 3 index contractions, so still gives a $(-1)^3 = -1$ sign

[5]Also recall that even for $s = 0$ one still has $u = 4m^2$ in the forward limit. So at least one of the Mandelstam variable will always be positive if two of the other ones are zero. In order to have all Mandelstam variables set to zero, one necessarily requires $m = 0$. Nonetheless this is still fine because the positivity bounds should still hold for $m = 0$, $s = 0$, and this is where the field redefinition works and reproduces the same coefficients as in the bound.

where only terms up to $O(\phi^4)$ were kept. Then integrating by parts and neglecting any boundary terms allows to rewrite

$$\partial_\mu\phi(\partial\phi)^2\partial^\mu(\partial\phi)^2 = -(\partial\phi)^2\Box(\partial\phi)^2$$

and

$$(\partial_\mu\phi)\partial^\mu(\partial\phi)^2 = -\Box\phi(\partial\phi)^2$$

The cubic term transforms as

$$\frac{g3}{\Lambda^3}(\partial\phi)^2\Box\phi \to \frac{g_3}{\Lambda^3}(\partial\phi)^2\Box\phi + \frac{2g_3^2}{\Lambda^6}(\partial_\mu\phi)(\partial^\mu(\partial\phi)^2)\Box\phi + \frac{g_3^2}{\Lambda^6}(\phi)^2\Box(\phi)^2 \tag{3.27}$$

Now the first term in here is exactly cancelled by a term of the same form in the kinetic part[6]. Further integration of the remaining expressions in the kinetic and cubic terms then result in

$$L \to -\frac{1}{2}(\partial\phi)^2 + (g_4 - g^2 - 3)(\partial\phi)^2((\Box\phi)^2 - (\partial_{\mu\nu})^2) \tag{3.28}$$

or in terms of a new coupling constant $g_4' = g_3^2 - g_4$

$$L \to -\frac{1}{2}(\partial\phi)^2 - g_4'(\partial\phi)^2((\Box\phi)^2 - (\partial_{\mu\nu})^2) \tag{3.29}$$

This has now showed that the cubic term is not independent of the quartic interaction, and also the superposition $g_4' = g_3^2 - g_4$ has the same form of $ag_3^2 + bg_4$ with $a = 1$ and $b = -1$ as in the positivity bound $g_3^2 - g_4 < 0$, confirming the coefficients in the bound above.

Note that if the metric $(- + ++)$ is used this result of course still holds, however, then the Lagrangian is has opposite sing in the kinetic and quartic terms and the field redefinition's second term also has a negative sign,

yielding the new coupling constant $g_4' = g_4 + g_3^2$, where now $a = b = 1$, as in the $(- + ++)$ positivity bound $g_3^2 + g_4 > 0$ above.

I therefore proved that the result from the amplitude calculations are correct and agree with field redefinition. Also it is shown that the cubic term can in fact be absorbed into the quartic one. The fact that the cubic term is not independent has important consequences for the EFT because, as explained in chapter 3.1 as well as in chapter 2.3, quartic EFT vertices correspond to cubic UV vertices, so the calculation above directly shows the significance of such absorptions.

[6] Recall that the kinetic term gets multiplied by a factor of $\frac{1}{2}$ so the factors should be the same but with the opposite sign

Chapter 4

Vector bounds

4.1 Proca bounds

In the following, Proca vectors, that is massive spin=1 vector fields, shall be reviewed first. Assuming real vector fields A_μ and defining the Maxwell tensor as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, one can write the Proca Lagrangian density as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu.$$

From this it is trivial to show that the equations of motion corresponding to the Euler-Lagrange-equations are

$$\partial_\mu \Big(\partial^\mu A^\nu - \partial^\nu A \mu \Big) + m^2 A^\nu = 0,$$

or

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

This is known as the Proca equation[17]. Applying ∂_ν on both sides gives

$$\partial_\nu \Big(\partial_\mu F^{\mu\nu} + m^2 A^\nu \Big) = 0$$

The first term is clearly a contraction of a symmetric and an antisymmetric tensor and is therefore zero. Hence this implies $m^2 \partial_\nu A^\nu = 0$ or

$$\partial_\nu A^\nu = 0.$$

This the general Lorentz gauge condition for massive vectors. Note that this was generally proven, which means that this not a gauge choice, but actually a necessity for massive vectors, implying there is no gauge freedom for Proca fields any more (unlike for massless photons).

Now substituting this Lorentz gauge condition back into the Proca equation yields

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu + m^2 A^\nu = 0$$

$$\Leftrightarrow \left(\Box + m^2\right) A^\nu = 0,$$

i.e. a Klein-Gordon-equation for 4-vectors. Hence the Proca equation reduces to 4 separate Klein-Gordon equations. This makes sense as we can decompose a 4-vector, being a 4 dimensional tensor of rank 1, as $4 = 3 \bigoplus 1$, i.e. a direct sum of a 3 dimensional Euclidean vector and a 1 dimensional scalar. This decomposition is trivial as long as A^ν transforms under the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ in the usual way, which is satisfied here of course as A^ν has been treated as a 4 vector.

In order to motivate this, one can consider the the Stückelberg action

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial^\mu \phi + m A'^\mu)(\partial_\mu + m A'_\mu) \tag{4.1}$$

for a massless vector field A'_μ. The well-known [18] Stückelberg trick now involves showing the correspondence between the Stückelberg Lagrangian and the Proca Lagrangian.

On the one hand one can either start with a massive vector field A_μ described by the Proca action, take its helicity-0 mode ϕ and use a transformation of the form

$$A_\mu \to \partial_\mu \phi + m A'_\mu,$$

which when plugged into the Proca action directly gives the Stückelberg action. This transformation is *not* a gauge transformation in the usual sense because here clearly ϕ and A_μ belong to the *same* field, opposed to the standard gauge transformation that couples a scalar field ϕ to an independent gauge field A'_μ. Nonetheless, it shall be stated that at least qualitatively the factor m does appear to act in a similar way as a usual coupling constant since by acting with m on a massless A'_μ, one receives a massive vector A_μ, which is clearly a type of Higgs mechanism giving mass to the initially massless A'_μ. Hence it is conventionally customary [18] to write the transformation[1] like $A_\mu \to \mathrm{D}_\mu \phi = \partial_\mu \phi + m A'_\mu$, despite it not being an actual covariant derivative.

On the other hand, starting with the Stückelberg action and realising it describes a massless field A'_μ means there must be some gauge freedom due to the nature of the quantisation of a massless vector field. This usually implies the Lorentz gauge condition $\partial^\mu A_\mu = 0$, however, unitary gauge for the helicity-0 mode can be chosen as well, that is $\partial_\mu \phi = 0$ Then simply fixing the gauge in the Stückelberg action by setting $\partial_\mu \phi = 0$ immediately gives the Proca action. This equivalence is remarkable because it means massive vectors can be described in terms of massless vectors. In a broader picture, this correspondence also includes the the fact that massive vectors have 2 transverse polarisation bases, giving 2 degrees of freedom due to gauge freedom of the massless A'_μ, whereas the massive A_μ does not have any residual gauge freedom any more. This is beautifully manifested in the Stückelberg trick simply by showing that the reason for the difference in the gauge freedom for each case is by *explicitly choosing a gauge*, which gets rid of the residual gauge freedom when going from A'_μ to A_μ.

[1] sometimes [15] it is even written as $\phi_\mu = \mathrm{D}_\mu$

The Lagrangians above are for the full UV complete theory, for low-energy limits however, one requires an EFT version of these actions that directly shows the scale dependence on a scale Λ. If one were to start with the Stückelberg action, and added operators of higher dimension, then to quartic order [15]

$$
\begin{aligned}
g_*^2 L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\mathrm{D}_\mu\phi)(\mathrm{D}^\mu\phi) + \frac{g_0}{\Lambda_\phi^4}(D_\mu\phi)^4 + \frac{g'}{\Lambda_\phi^3}(\partial_\mu D_\nu\phi)(D^\nu\phi D^\mu\phi) \\
+\frac{1}{\Lambda_\phi^6}\left(g_1 D_\mu\phi\partial^\mu D^\nu\phi + g_2\partial_\mu D_\nu\phi D^\nu\phi + g_3 D_\mu\phi D^\mu\phi\partial_\alpha\phi_\beta\partial^\beta\phi^\alpha)^2\right) \\
+\frac{1}{\Lambda_A^4}\left(g_4 F_{\mu\nu}F^{\rho\nu}F_{\sigma\rho}F^{\mu\sigma} + g_5(F_{\mu\nu}F^{\mu\nu})^2\right) \\
+\frac{m^4}{\Lambda_\phi^6}\left(g_6 D_\mu\phi D^\nu\phi F^{\alpha\mu}F_{\alpha\nu} + g_7(D_\mu\phi)^2(F_{\mu\nu})^2\right)
\end{aligned}
\tag{4.2}
$$

Clearly for the Stückelberg version, A_μ and ϕ are *effectively* treated as different fields, so it is sensible to give each mode its own mass scale, i.e. Λ_A for A'_μ and Λ_ϕ for ϕ. It is also clear from dimensional grounds that

$$
\Lambda_\phi^3 = m\Lambda_A^2
$$

Consider this Lagrangian:

$$
\begin{aligned}
g_*^2 L_{\mathrm{EFT}} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{L_1} \underbrace{-\frac{1}{2}m^2 A_\mu A^\mu}_{L_2} + \underbrace{\frac{m^4 g_0}{\Lambda_\phi^4}(A_\mu A^\mu)^2}_{L_3} \\
+\frac{m^4}{\Lambda_\phi^6}\Big(\underbrace{g_2 A_\mu A_\nu\partial^\mu A_\rho\partial^\nu A^\rho}_{L_4} + \underbrace{g_2 A_\mu A_\nu\partial_\rho A^\mu\partial^\rho A^\nu}_{L_5} + \underbrace{g_3 A_\mu A^\mu\partial_\alpha A_\beta\partial^\beta A^\alpha}_{L_6}\Big) \\
+\frac{1}{\Lambda_A^4}\Big(\underbrace{g_4 F_{\mu\nu}F^{\rho\nu}F_{\rho\sigma}F^{\mu\sigma}}_{L_7} + \underbrace{g_5 F_{\mu\nu}F^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}}_{L_8}\Big) + \frac{m^4}{\Lambda_\phi^6}\Big(\underbrace{g_6 A_\mu A^\nu F^{\alpha\mu}F_{\alpha\nu}}_{L_9} + \underbrace{g_7 F_{\mu\nu}F^{\mu\nu}A_\alpha A\alpha}_{L_{10}}\Big)
\end{aligned}
\tag{4.3}
$$

Here we defined the coupling constants as $g_0 = a_0, g_1 = a_3, g_2 = a_4, g_3 = a_5, g_4 = c_1, g_5 = c_2, g_6 = C_1$ and $g_7 = C_2$ to get the Lagrangian of [**?**].

4.1.1 polarisation contractions

As there is no gauge freedom any more cant set any of the 4 to zero but 2 are still dependent on each other, usually taken to be scalar and longitudinal mode.

For any given particle the coordinate system can be chosen so that the z-axis aligns with the total 3-momentum vector. Then the particle's 4-momentum is given by

$$
p_\mu = (E, 0, 0, p)
$$

Given that for the Proca vector field there is no gauge freedom any more, all 4 spacetime components are non-zero in general. However, there are still 3 degrees of freedom, and usually the 2 modes taken to be dependent on each other are the scalar and longitudinal modes. Therefore we are allowed to define 3 linearly independent polarisation bases, constructing all 4-dimensional polarisations. For instance, we may take

$$\epsilon_\mu^1 = (0, 1, 0, 0),$$

$$\epsilon_\mu^2 = (0, 0, 1, 0)$$

$$\epsilon_\mu^3 = \frac{1}{m}(p, 0, 0, E).$$

From the Lorentz condition we expect the momentum to be orthogonal to all polarisation vectors, and explicit calculation verifies that

$$p_\mu \epsilon^{i,\mu} = 0 \; \forall i \in \{1, 2, 3\}$$

for the basis above. The total polarisation can thus be expressed as a superposition of the basis vectors as

$$p_\mu = ap_\mu^1 + bp_\mu^2 + cp_\mu^3 = \left(\frac{c}{m}p, a, b, \frac{c}{m}E\right)$$

with parameters a, b and c. The usual normalisation condition requires

$$|a|^2 + |b|^2 + |c|^2 = 1$$

which allows to express one parameter in terms of the other 2.

momentum-momentum contraction

All momentum contractions can be trivially expressed in terms of Mandelstam variables using the equations in chapter 2.

polarisation-momentum contraction

Contractions of momenta with polarisations can be simplified in the following way. For contractions of the form $(p_i \cdot \epsilon_j)$ we have to distinguish two cases:

1. The polarisation and momentum vectors lie on the same side of the scattering, that is, both ingoing or both outgoing. In other words $i, j \in \{1, 2\}$ OR $i, j \in \{3, 4\}$. Now with the elastic condition we have in the CoM frame $\vec{p_i} = -\vec{p_j}$ and $E_i = E_j$. This means we can express s as

$$s = (p_{i,\mu} + p_{j,\mu})^2 = \left(\begin{pmatrix} E_i \\ 0 \\ 0 \\ p_i \end{pmatrix} + \begin{pmatrix} E_j \\ 0 \\ 0 \\ p_j \end{pmatrix} \right)^2 = \left(\begin{pmatrix} E_i + E_j \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)^2 = 4E_i^2 = 4E_j^2$$

This implies that $E_i = E_j = \frac{\sqrt{s}}{2}$ Now evaluating $(p_i \cdot \epsilon_j)$ immediately gives

$$(p_i \cdot \epsilon_j) = \frac{c_j}{m} p_j E_i - \frac{c_j}{m} p_i E_j = \frac{c_j}{m}(p_j E_i - p_i E_j) = \frac{2c_j}{m} p_j E_j = \frac{c_j}{m} p_j \sqrt{s}$$

Now we need to express p_j in terms of s. We know that $E_j^2 = m^2 + \vec{p}^2 = m^2 + p_j^2$ and $E_j^2 = \frac{s}{4}$, so therefore $\frac{s}{4} = m^2 + p_1^2$, which is

$$p_j = \sqrt{\frac{s}{4} - m^2}$$

after rearranging. Note that of course in the range of $s < 4m^2$, corresponding to the unphysical region, we have a non-real value, however, this is fine since we allow amplitudes to be complex and take the modulus squared which is real and positive. Putting everything together yields

$$(p_i \cdot \epsilon_j) = \frac{c_j}{m} \sqrt{\frac{s^2}{4} - m^2 s}$$

It is important to see that the root is entirely given in terms of s and the polarisation constant c_j has the label of the polarisation vector, not of the momentum vector.

2. The polarisation and momentum vectors lie on different sides of the scattering, that is, one ingoing and one outgoing. In other words, one of them is labelled by $\{1, 2\}$ and the other is labelled by $\{3, 4\}$. For this case we need to invoke the second assumption on only focusing on the forward scattering limit. This then additionally gives

$$t = (p_{1,\mu} - p_{3,\mu})^2 = \left(\begin{pmatrix} E_1 \\ 0 \\ 0 \\ p_1 \end{pmatrix} - \begin{pmatrix} E_3 \\ 0 \\ 0 \\ p_3 \end{pmatrix} \right)^2 = 0,$$

implying that the 4-vectors have to be the same i.e. $p_{1,\mu} = p_{3,\mu}$, so that the energies and linear momenta are the same for particles 1 and 3, so $E_1 = E_3$ and $p_1 = p_3$ (and likewise for particles 2 and 4). Invoking the usual elastic conditions for the CoM A) $p_1 = -p_2$, B) $p_3 = -p_4$, C) $E_1 = E_2$ and D) $E_3 = E_4$ then results in

$$p_1 = p_3 = -p_2 = -p_4$$

and

$$E_1 = E_2 = E_3 = E_4.$$

From these it is now clear that there is a crossing between incoming and outcoming particle momenta, which we did not have in case 1.

Hence, for $p_i = -p_j$ with i, j belonging to *different* crossing sides we can then summarise

$$(p_i \cdot \epsilon_j) = \frac{c_j}{m}(p_j E_i - p_i E_j) = \frac{2c_j}{m} p_j E_j = \frac{c_j}{m}\sqrt{\frac{s^2}{4} - m^2 s}$$

which is the same result as above. It is not trivial to see that these two cases are in fact the same because in case 1 we exploit symmetry in the centre of mass frame, however, if we consider a contraction between a particle that has already scattered and a particle that will scatter e.g. particle 1 and particle 3, then we no longer have an obvious choice for the CoM frame because those particles will only exist at different times. Now when taking the forward scattering limit, we make the choice that all four particles have the same energies and their momenta may only differ by a sign, so this now ensures that a contraction between e.g. particle 1 and particle 3 gives the same kinematics as a contraction between particle 1 and particle 2, given by case 1.

Note that here we still require $i \neq j$ because otherwise the contraction is directly zero due to orthogonality, as demonstrated above.

polarisation-polarisation contraction

Focusing on the last possible contraction for vector fields, we now investigate contractions of the form $\epsilon_i \cdot \epsilon_j$. This is rather straight forward, despite some minor subtleties. We begin by explicitly showing that

$$\begin{aligned}
\epsilon_i \cdot \epsilon_j &= \begin{pmatrix} \frac{c_i}{m}p_i & a_i & b_i & \frac{c_i}{m}E_i \end{pmatrix} \begin{pmatrix} \frac{c_j}{m}p_j \\ -a_j \\ -b_j \\ -\frac{c_j}{m}E_j \end{pmatrix} \qquad (4.4) \\
&= \frac{c_i c_j}{m^2} p_i p_j - a_i a_j - b_i b_j - \frac{c_i c_j}{m^2} E_i E_j \\
&= -\frac{c_i c_j}{m^2}(p_i \cdot p_j) - a_i a_j - b_i b_j \\
&= -\frac{c_i c_j}{m^2}(p_i \cdot p_j) - \xi_{i,j}
\end{aligned}$$

where for simplicity the abbreviation $\xi_{i,j} = a_i a_j + b_i b_j$ is used and we recalled $(p_i \cdot p_j) := p_{i,\mu} p_j^\mu = E_i E_j - p_i p_j$.

4.1.2 Significance of polarisation

It is important to consider the question as to why the polarisation parameters are necessary to be of this general form, and cannot be simplified further. We know that due to conservation of momentum we can always express functions, that are written in terms of sums or products of momenta pairs, in terms of complimentary momenta pairs. For instance $(p_1 \cdot p_2) = (p_3 \cdot p_4)$ and $p_{1,\mu} + p_{2,\mu} = p_{3,\mu} + p_{4,\mu}$ so

$$f(p_1 p_2) = f(p_3 p_4)$$

All we have to do is focus on two particles and we can infer the kinematics of the other directly from the first consideration.

Given this, one could consider the same scenario for the polarisation parameters above and finds that such simplifications do not exist. There is no physical or mathematical reason why expressions like $c_1 + c_2 = c_3 + c_4$ should hold in general, opposing the naive initial guess that complimentary pairs of parameters exist for every quantity. This is naturally related to the fact that polarisations (more specifically the basis of polarisation) of particles are not conserved. We can always pick out an initial polarisation basis for the two incoming particles by choosing specific values for the corresponding a,b and c. Experimentally this would be done by adding polarisers to the scattering experiments, selecting particles with only a specific polarisation basis. However, the entire point of QFT is that an interaction may (and will) create every type of particles described by the Lagrangian, which is why the plethora of particles produced will include every single particle type and one usually sums over all particle states, including all polarisation states. That means that after the scattering new particles with different polarisation will appear, so the initial choice of the basis and using a polariser does not have an effect on the particle spectrum at all. This implies that the polarisation on its own is not an invariant quantity. Using a polariser for incoming particles would only be relevant when one uses a second polariser for outgoing particles. Then, and only then, will a specific subset of states be measured, corresponding to the amplitude. This is referred to as definite polarisation, where the polarisations of all particles are known. Of course in such a case one picks specific numbers for a,b and c for calculations so there are no degrees of freedom left. Opposed to this is the concept of indefinite polarisation, where no polariser experimentally picks out any specific basis, all states are summed and averaged over and in calculations no choice is made for a,b,c experimentally. Therefore, for 2-2 scattering, one ends up with 20 degrees of freedom corresponding to the 20 parameters that are left after considering the $3 \cdot 4 = 12$ possible complex parameters a, b and c for each particle (so 24 real dofs) and subtracting the four normalisation conditions. One can even refer to classical Electromagnetism, admitting circular and elliptical polarisations that are evolving with time. This is another example of how polarisation is not invariant with respect to time without having to analyse scattering process altogether. We can summarise this discussion by the simple rule *There is no conservation of polarisation in scattering*

4.1.3 EFT amplitude

It is now the goal to explicitly calculate the amplitude of the Proca EFT - a lengthy process that requires a lot of bookwork.

Starting with the first term L_1 we consider two separate, free propagator and all its permutations. This yields

$$A_1 = -c_1c_2(\frac{s}{2} - m^2 - \xi_{12} - c_3c_4(\frac{s}{2} - m^2) - \xi_{34})$$

$$+(2\frac{c_2c_1 + c_3c_4}{m^2} + 2\frac{c_1c_3 + c_2c_4}{m^2} + 2\frac{c_1c_4 + c_3c_2}{m^2})(\frac{s^2}{4} - m^2 s)$$

The first line is of order $O(s)$ and will therefore vanish in the second derivatives, so only the second line is relevant.

For L_2 we also consider 2 free propagators and see that the result is

$$A_2 = (c_1c_4 + c_2c_3)(2m^2 - u) + (c_1c_3 + c_4c_2)(2m^2 - t) + (c_1c_2 + c_3c_4)(s - 2m^2) + 2m^2\Sigma_{i,j}\xi_{i,j}$$

For L_3: one has 2 polarisation contractions such as

$$A_3^s = \frac{8m^4 g_0}{\Lambda_\phi^4}(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})(-\frac{c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})$$

For L_4 one has one polarisation-momentum contraction and one polarisation-polarisation contraction

$$A_4^s = \frac{8g_1m^2}{\Lambda_\phi^6}c_1c_2(\frac{s^2}{4} - m^2 s)(\frac{-c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})$$

For L_5 one has 2 polarisation contractions and an additional momentum-momentum contraction

$$A_5^s = \frac{8g_2m^4}{\Lambda_\phi^6}(m^2 - \frac{t}{2})(\frac{-c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

For L_6 one has two polarisation-momentum contractions and one polarisation-polarisation one

$$A_6^s = \frac{8g_3m^2}{\Lambda_\phi^6}c_3c_4(\frac{s^2}{4} - m^2 s)(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

For L_{7a} there are 16 different cases, which are all stated in the appendix.

For L_{8b} one has 4 polarisation-momentum contractions

$$A_{8b} = A_{8b} = A_{8b}^s + A_{8b}^t + A_{8b}^u = \frac{96g_5}{\Lambda_A^4}c_1c_2c_3c_4(\frac{s^2}{4} - m^2 s)^2$$

L_9 and L_{10} both have many combinations as well and are listed.

As the Lagrangian we are interested in is

$$L_{\text{EFT}} = \Sigma_{i\in\{1,2,\ldots,9,10\}}L_i$$

we now sum over all of these amplitudes in every channel and get

$$M = \Sigma_{j=s,t,u}\Sigma_{i\in\{1_j,\ldots,10_j\}}M_i \tag{4.5}$$

The appendix explicitly lists the result for each channel and each term.

4.1.4 Positivity bound

Differentiating all expressions for the amplitudes above and in the appendix is cumbersome, but straight forward, and the results are listed in the appendix. Then invoking the positivity bound of lowest order yields the following results.

Due to linearity

$$\frac{\partial^2 M_{EFT}}{\partial s^2} = \frac{\partial^2 \Sigma_i M_i}{\partial s^2} = \Sigma_i \frac{\partial^2 M_i}{\partial s^2} > 0$$

so all individual terms above have to be added and their sum is positive.

This is achieved by setting $t = 0$ in the forward limit. In addition, since the bound has to hold for any s, it also has to be true for exactly $s = 0$, which significantly simplifies the bounds above.

This is the most general result for indefinite polarisations.

Definite polarisation bounds

One can use these results to also pick a specific basis by selecting the parameters with the 20 degrees of freedom. Now in the helicity basis, $+$ and $-$ are assigned to the two transverse bases, and 0 to the longitudinal mode. This notation is different from the basic helicity representation, where one usually picks out a specific spin and projects this spin *onto* the momentum, giving $+/-$ states corresponding to right-handed and left-handed states. Here, however, we label the transverse states as $+/-$. So these two notations are the same:

$$\langle 1| = \alpha_+ \langle +1| + \alpha_0 \langle 0| + \alpha_- \langle -1|$$

and

$$\epsilon_1 = a \cdot \epsilon_{\text{transverse1}} + b \cdot \epsilon_{\text{transverse2}} + c \cdot \epsilon_{\text{longitudinal}}$$

Now picking any specific a_i, b_i, c_i gives a definite bound of the form

$$n_0 \frac{g_0}{\Lambda_\phi^6} + n_1 \frac{g_1}{\Lambda_\phi^6} + n_2 \frac{g_2}{\Lambda_\phi^6} + n_3 \frac{g_3}{\Lambda_\phi^6} + n_4 \frac{g_4}{\Lambda_A^4} + n_5 \frac{g_5}{\Lambda_A^4} + n_6 \frac{g_6}{\Lambda_\phi^6} + n_7 \frac{g_7}{\Lambda_\phi^4} > 0.$$

There infinitely many definite positivity bounds, but choosing methodically which values to pick and polarisation states to consider greatly reduces the set of physically distinct amplitudes.

One can then consider incoming and outgoing states that entirely lie along one of the three polarisation states. For 2-2 scattering, this then covers 4 particles, with 3 polarisation possibilities $(+/0/-)$ each, i.e. 12 different processes. However, due to crossing symmetry there is a further redundancy and only the following 4 bases will be evaluated.

For $(++++) = (++ \to ++)$ then the incoming and outgoing particles are in the $|++\rangle$ state, so

$$\forall i \in \{1, 2, 3, 4\} : a_i = 1, b_i = 0$$

and

$$c_i = 0.$$

This sets any terms that are mutliplied by any c_i to zero and sets all ξ_{ij} to 1. Now it is apparent that physically, (- - - -) should give the same result as this would set the b_i to 1 and a_i to 0, but both transverse directions are equivalent and indistinguishable, so focusing on (++++) is sufficient. Plugging in these parameter choices into the indefinite bounds yields

$$-4\frac{g_3}{\Lambda_\phi^6} - 96\frac{g_5}{\Lambda_A^4} - 16\frac{g_6}{\Lambda_\phi^6} > 0$$

Assuming $\Lambda_A^2 >> \frac{\Lambda_\phi^3}{m}$, this then reduces to

$$-4\frac{g_3}{\Lambda_\phi^4} > 0 \leftrightarrow g_3 < 0.$$

For (0000) all particles are entirely polarised along the momentum direction, i.e. along the longitudinal basis, so

$$\forall i \in \{1, 2, 3, 4\} : a_i = b_i = 0$$

and

$$c_i = 1.$$

This implies $\xi_{ij} = 0$ and the definite bound is then

$$8\frac{g_0}{\Lambda_\phi^6} + 40\frac{g_1}{\Lambda_\phi^6} - 24\frac{g_2}{\Lambda_\phi^6} + 28\frac{g_3}{\Lambda_\phi^6} + 12\frac{g_6}{\Lambda_\phi^6} + 68\frac{g_7}{\Lambda_\phi^6} > 0$$

For $(- + -+)$ the particles have opposite transverse polarisation without any longitudinal components, giving

$$\forall i \in \{1, 2, 3, 4\} : a_1 = b_2 = a_3 = b_4 = 0,$$

$$a_2 = b_1 = a_4 = b_3 = 1$$

and

$$c_i = 0.$$

This is equivalent to setting all c_i to zero and $\xi_{12} = \xi_{14} = 0, \xi_{13} = 1$, so effectively the only terms that do not vanish correspond to a subset of the t-channels. The bound is

$$-4\frac{g_3 m^2}{\Lambda_\phi^6} - 4\frac{g_4}{\Lambda_A^4} > 0$$

or

$$g_4 < 0$$

For $(0+0+)$ we similarly have

$$\forall i \in \{1,2,3,4\} : a_1 = c_2 = a_3 = c_4 = 0,$$

$$a_2 = c_1 = a_4 = c_3 = 1$$

and

$$b_i = 0.$$

Hence the only terms that do not vanish are the ones that contain ξ_{24} and c_1, c_3 and the bound is the same as above for

$$g_4 < 0$$

so in this particular case two bounds are accidentally redundant.

Naturally these bounds are quite weak, but this is because the elastic, forward limit was chosen, although this still allows to impose certain properties on coupling constants. For stronger bounds, probing beyond the forward will have to be implemented, as done in [15]. Recalling that $g_3 = a_5$ and $g_4 = c_1$ was defined, one can compare the results above to this paper, which implies the general bounds in the forward limit are $g_0 > 0$ and $g_1 + g_6 > 0$ when assuming $g_4 << 1$. The (-+-+) bound above gives $g_4 < 0$, so agrees with $g_4 << 1$ The (0000) bound above gives $g_0 > 0$ when the couplings g_1 to g_7 are turned off, and the (0000) bound also gives $40g_1 + 12g_6 > 0$ when all other couplings are turned off, or $\frac{10}{3}g_1 + g_6 > 0$. This is a slightly weaker statement of $g_1 + g_6 > 0$ as long as $g_6 < 0$. Hence the positivity bounds agree for $g_6 < 0$.

It shall be pointed out that any bounds determining the sign of one coupling constant reduce the parameter space by 50%. Hence knowing that $g_3 < 0$ and $g_4 < 0$ is reduces the parameter space by a factor of 4, or 75%, showing the significance of such positivity considerations.

Chapter 5

Electroweak bounds for Quartic Vector Boson Scattering

I will now apply these results to SM EFTs. While there are 18 different dimension-8 operators contributing to the scattering processes [19], calculating all bounds for all Wilson coefficients for the 18 operators with indefinite bases is beyond the scope of this project. Instead I will focus on a limited subset of specific operators with definite polarisation states, which will still greatly reduce the parameter space. In the following, the methods and conventions of [19], [21], [22] and [23] will be used.

5.1 Standard Model as an EFT

Historically, the Standard Model has always been described by a Lagrangian that is a collection of terms obeying the required and observed symmetries. The SM Lagrangian therefore represents a brute-force method of creating a universal model that sums up all the different interactions possible. By definition, all terms admitted to the Standard Model have to be experimentally verifiable, which can only be tested up to an associated scale Λ_i. This means that all SM operators have an intrinsic scale at which their behaviour becomes important and the Standard Model can be written as

$$L_{SM} = \sum_i \sum_{d=0}^{4} c_i \hat{O}_{i,d} = \sum_i \sum_{d=0}^{4} \frac{c_i'}{\Lambda_i^{d-4}} \hat{O}_{i,d}$$

and the Wilson coefficients act as effective coupling constants for each term. The direct scale dependence Λ_i for each term encourages the EFT nature of the SM. This means the SM is not complete, and so far mostly operators up to dimension 4 have been accessible. Historically the terms in the SM EFT have changed with every new experimental discovery that required new theoretical guidance.

For instance, the Dirac Lagrangian is applicable for Dirac fermions and therefore an early model of the SM directly included such a Dirac term. However, as most leptons are left-handed except for right-handed neutrinos, the Dirac term in the SM

Lagrangian must be replaced by terms that separately treat left- and right-handed elementary particles.

This is exactly how EFTs work; they theoretically correspond to a low energy approximation of a UV complete theory, however, in practise they simply allow experiments to be conducted up to the scale (eg Λ_1) of a given collider, where the mass scale Λ_{m_1} of the particles produced in such colliders is $\Lambda_1 \geq \Lambda_{m_1}$). Every time a new collider upgrade reaches higher energies (eg Λ_2), (hopefully) new particles with mass scale Λ_{m_2} get created that were inaccessible before. This then naturally describes a hierarchy

$$\Lambda_2 \gg \Lambda_{m_2} \gg \Lambda_1 \gg \Lambda_{m_1}$$

etc that can only be decoded step by step.

If such discoveries are then made, new terms with operators up to Λ_{m_2} are then added to the Standard Model[1]

This then allows to introduce the following schematics for an SM EFT:

1. Choosing the desired particle spectrum by selecting a specific energy cut-off Λ
2. Integrating out all particles above Λ
3. Choosing the vertex order (number of vertices) to be considered
4. Calculating amplitude (and possible positivity bounds)

An example for this is on how this is exactly implemented in the Standard Model is for e.g. the muon decay $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. In the SM this weak decay is mediated by the exchange of a weak boson W^-, and the corresponding terms in the SM Lagrangian are [24]

$$L_{SM} \supset \frac{g}{2\sqrt{2}}(J^\mu W_\mu^+ + J^{\mu\dagger} W_\mu^-)$$

where J_μ is the charged lepton current. Here the mass of the mediator is related to the decay time as well as the Feynman propagator as it is a virtual particle. In any case, it is clear that the mass of the W^- vector boson m_W sets the scale of the scattering. Hence if one then considers a model that has a cut off below m_W, then integrating out this gauge boson results in the new Lagrangian

$$L_{EFT} \supset -\frac{G_F}{\sqrt{2}} J^{\mu\dagger} J_\mu$$

with the Fermi constant G_F and the new EFT Lagrangian is now independent of the $W^\pm$. Also $G_F \propto \frac{g^2}{m_W^2}$ so the new EFT coupling constant (i.e. Wilson coefficient) G_F is

[1]technically the boundary between what is SM and what is BSM is very thin, if new particles are detected and then added to the SM Lagrangian one can argue that either the SM got extended, or the SM simply got replaced by another BSM model. Both interpretations should be equivalent.

now expressed in terms of a UV coefficient g, and for m_W much bigger than the EFT cut-off, $\frac{g^2}{m_W^2} \ll 1$ and hence $G_F \ll 1$ This is a low energy approximation of the full SM Lagrangian. It is then apparent that similar arguments can be made to call the SM in its current form an EFT, which only includes particles with mass up to the maximum SM scale.

5.2 Beyond Standard Model EFT for Vector Boson Scattering

For the scope of this project only the Higgs and electroweak sector shall be focused on, so the hadronic sector and QCD will be neglected in the investigated scattering. Vector Boson Scattering (VBS) plays an important role in understanding and confirming scattering theory and searches for new physics [25].

It is for this reason that the for VBS relevant part of the Standard Model L_{SM} is added to the higher dimensional, Beyond-the-Standard-Model (BSMEFT) operators. Without loss of generality, one can therefore write the Lagrangian to be investigated as

$$L_{BSM-EFT} = L_{SM} + \sum_d \sum_i c_{i,d} \frac{\hat{O}_{i,d}}{\Lambda^{d-4}}, \tag{5.1}$$

where $\hat{O}_{i,d}$ are d dimensional operators with the required BSM properties and $c_{i,d}$ are the corresponding Wilsonian coefficients in the EFT expansion. Here i labels the number of the operator, so here for 18 different dimension-8 operators the labels very clearly are within $1 \leq i \leq 18$ and $d = 8$.

Vector Boson Scattering is due to the interaction of vector gauge bosons such as the $W^{\pm}$, the Z^0 and the photon γ as well as the interaction of these vector bosons with scalars, as the Higgs field ϕ. For vector-vector interactions the Standard Model Lagrangian allows two types of interactions: cubic vertices, also known as Tri Gauge Coupling (TGC) and quartic vertices, also known as Quartic Gauge Coupling (QGC). It turns out that cubic vector interactions can be constrained with WW measurements [19], so focusing on the QGC diagrams is sufficient at this level. It shall also be mentioned that it is very significant that for QGC the vector fields are massive. This is due to the fact that gauge freedom is not residual any more for massive vectors, a fact that has become a theme throughout this dissertation. While gauge freedom prevents massless vector fields like photons from having terms in the Lagrangian like $m^2 A_\mu A^\mu$, for massive ones such terms are allowed. This allows the more complex QGC vertices that do not violate gauge.

In addition, dimension-6 operators contribute to the total amplitudes too, but their anomalous BSM behaviour can be experimentally measured in different reactions, from which dimension-6 characteristics can be inferred and the VBS correspondingly corrected [19] so that only the dimension-8 operators looked at in this dissertation are relevant.

The motivation for a higher-rank Higgs field is due to the fact that the discovery of the Higgs boson manifested fundamental assumptions of the Standard Model and understanding how particles acquire mass, nevertheless this does still not resolve all research questions in particle physics. For instance, given that the electroweak sector is determined by a $SU(2)_L \times U(1)_Y$ gauge symmetry (and its spontaneous symmetry breaking), one might use a general approach utilising a 2×2 matrix representation $H$for the Higgs field. Another way of representing this would be a 2×1 complex column vector $\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$, which has 2 complex, i.e. 4 real components and therefore 4 degrees of freedom. Clearly representing these 4 components via a SU(2) doublet (the column vector representation) or the direct representation as 4 entries in the Higgs matrix is equivalent. Explicitly the Higgs matrix is

$$H = \frac{1}{2} \begin{pmatrix} v + t - iw^3 & -i\sqrt{2}w^+ \\ -i\sqrt{2}w^- & v + h + iw^3 \end{pmatrix}$$

and transforms linearly under SU(2) symmetries, as required[21]. That is

$$H \to XHY^\dagger$$

with $X \in SU(2)_L$ and $y \in SU(2)_R$.

SU(2) has $n^2 - 1 = 2^2 - 1 = 3$ degrees of freedom, i.e. 3 symmetry generators, and therefore requires 3 gauge fields W_μ^i with $i = 1, 2, 3$ to couple to the scalar field and make the Lagrangian invariant under SU(2) transformations by replacing ordinary partial derivatives with the covariant derivative $\partial_\mu \phi \to D_\mu \phi$. Similarly, the U(1) field has 1 degree of freedom, corresponding to global phase shift transformations, and hence only offers one generator. Hence the U(1) group only has one single vector field B_μ acting as a gauge field turning the global symmetry into a local one. As a result, SU(2)×U(1) has 4 generators, of which 3 give rise to the W_μ^i fields and one B_μ vector field. Physically, in the SM the W_μ vector gauge fields are the W^+, the W^- and the Z^0 vector gauge bosons, and the B_μ is the electromagnetic photon field.

This discussion of electroweak unification then implies that the total covariant derivative for the whole SU(2)×U(1) group then is

$$D_\mu \phi = \partial_\mu \phi - igW'_\mu \phi - ig' B_\mu \phi$$

with $W'_\mu = \frac{1}{2} W_\mu^i \tau^i$ where the τ^i are the SU(2) matrix generators.

and the SM Lagrangian is

$$L \supset (D_\mu \phi)^\dagger (D^\mu \phi) - \mu^2 \phi^\dagger \phi - h(\phi^\dagger \phi)^2$$

Naturally if the matrix representation is used, then the covariant derivative is

$$D_\mu H = \partial_\mu H - igW'_\mu H - ig' H B_\mu$$

and

$$L \supset -\frac{1}{2}\left[B_{\mu\nu}B^{\mu\nu}\right] - \frac{1}{2}\left[W_{\mu\nu}W^{\mu\nu}\right] + tr\left[(D_\mu\phi)^\dagger(D^\mu\phi)\right] + \mu^2 tr(H^\dagger H) - \frac{h}{2}tr(H^\dagger H)^2$$

where now the field strength tensors used are

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,$$

being the Maxwell field strength tensor, and

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig\left[W_\mu, W_\nu\right].$$

The literature has several sign conventions, [22] uses one with $-g$ and $-g'$ in the covariant derivative, whereas [19] makes use of $-g$ and $+g'$. Whichever one is chosen does not matter since the signs can always be absorbed into the coupling constants, however, this does have implications for any positivity bounds later on. In the following, [19]'s convention is chosen.

Whilst these gauge fields apply to the electroweak sector in the SM, for EFTs beyond the SM one adds operator structures with higher dimension to the SM Lagrangian as discussed above. The full list of dimension-8 operators is found in [19], and here S_0 is looked at in particular. Then

$$L \supset F_{s_0} Tr\left[(D_\mu H)^\dagger D_\nu H\right] Tr\left[(D^\mu H)^\dagger D^\nu H\right]$$

with Wilson coefficient f_{S_0} and there are 17 more operators that are similar. S_1 and S_2 have the same structure as S_1 except for permutations of the indices, giving slightly different contractions.

The Lagrangian for S_0 contains terms like

$$L \supset -g'^2 Tr\left[(HB_\nu)(HB^\nu)\right]\left[(B_\mu^\dagger H^\dagger)(B^{\mu\dagger}H^\dagger)\right]$$

here the trace sums over the indices of the 2 by 2 matrix H, so is not affecting the vector contractions. Then expressions like this can be simplified using

$$Tr\left[(HB_\nu)(HB^\nu)\right] = Tr\left[\sum_i (HB_i)(HB^i)\right] = \sum_i B_i B^i Tr\left[HH\right] = B_\mu B^\mu Tr\left[H^2\right]$$

Directly calculating $Tr\left[H^2\right]$ gives

$$Tr\left[H^2\right] = 2(v^2 + h^2 - (w^3)^2 + hv - 2w^+w^-)$$

Similar expressions are then obtained for $Tr\left[H^\dagger H\right]$ and $Tr\left[H^\dagger H^\dagger\right]$ These are the terms that are multiplied by any of the vector contractions of W_μ or B_μ. However, in the amplitude the vertices get additional factors only from momenta or polarisation contractions. For the vector contraction, the same frame work is used as in chapter

4, and for the momenta contractions, these stem from the partial derivative $\partial_\mu H$ in the covariant derivatives, each giving a Mandelstam variable of order O(s). S_0 has 4 covariant derivative terms containing 5 terms with 3 scattering channels, each with 5 polarisation parameters[2], giving 60×5 different cases one could consider.

Explicitly calculating for $L \supset -g'^2 Tr\left[(HB_\nu)(HB^\nu)\right]\left[(B_\mu^\dagger H^\dagger)(B^{\mu\dagger}H^\dagger)\right]$ and $(++++)$ the contribution to the positivity bound is then positive, i.e $24g'^2 > 0$ or

$$g' > 0$$

For $L \supset -g'^2 Tr\left[(\partial_\mu H)(\partial^\mu H)\right] Tr\left[(B_\mu^\dagger H^\dagger)(B^{\mu\dagger}H^\dagger)\right]$ and $(++)$ it is $16g'^2 > 0$ and one can similarly conduct this for all remaining terms in the Lagrangian to show the total bound is indeed positive[19]

$$g'^2 > 0$$

This then determines the sign of the overall coupling constant F_{S_0} for S_0, and similar analysis can be applied to S_1 and S_2.

For S_1 and S_2 their exclusion region is given by Fig. 5.1, comparing the bound to empiric results. Assuming a rescaling

$$\alpha_4 = \frac{v^4}{8\Lambda^4} f_{s_0}$$

and

$$\alpha_5 = \frac{v^4}{16\Lambda^4} f_{s_1}$$

then the bounds calculated in [19] imply the different exclusion zone in Fig. 5.2. Both plots imply that the allowed parameter space is greatly reduced since the overlap between the empiric results and the positivity bounds is very small. Hence, just like in chapter 4, positivity bounds have significant consequences for unitary EFTs.

[2] the 3 complex a,b,c minus 1 dof due to normalisation as in chapter 4

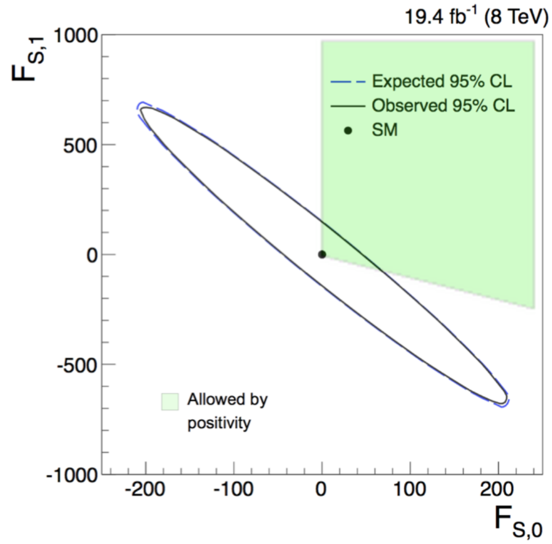

Figure 5.1: Reproduced from [19], this plot visualises the EFT exclusion zone for f.and f, showing the overlap between experimental results and theoretical constraints.

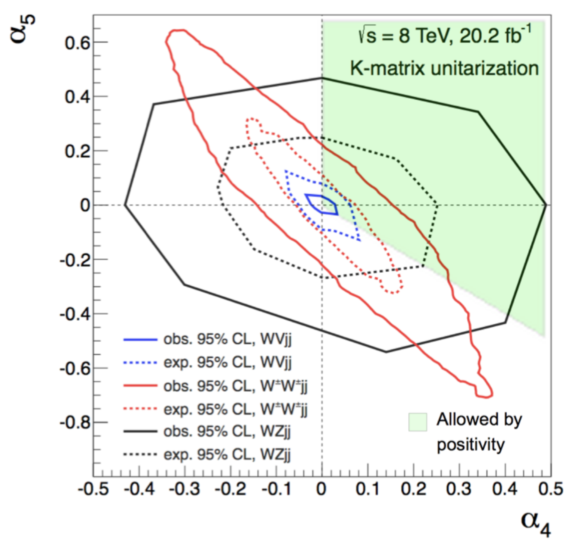

Figure 5.2: Reproduced from [19], here the bounds on α_4 and α_5 are shown and allow to infer that there is significant overlap between experimental data, including uncertainties due to measurement errors, and the positivity bounds on the superposition of coupling constants.

Chapter 6

Conclusion

Throughout this dissertation, multiple applications and direct calculations of positivity bounds for low energy EFTs were explicitly shown. For the scalar Lagrangian, the bound was shown to be $g_3^2 - g_4 < 0$ and confirmed by directly redefining the particle fields to absorb the cubic interactions into quartic ones. Being able to absorb the cubic terms showed that the coupling constants were not independent, and showed that the amplitude calculations, including the pole subtractions to receive finite polynomial terms in the amplitude, were consistent. The bounds associated with the Proca EFT were explicitly calculated and resulted in proving that the first order positivity constraints on the Wilsonian coefficients are $g_4 < 0$ and $g_5 < 0$. In order to do so, indefinite polarisation states have been considered, having yielded the most general form for these scattering bounds. Several finite states were then calculated to be in the elastic forward limit. The fact that these results are very different compared to massless vector fields was explained by the Stückelberg mechanism that is separately expressing fields in terms of their helicity-0 modes.

The methods derived for scalar and vector fields was then applied to VBS scattering, presenting a general approach for calculating SMEFT bounds for the 18 dimension-8 operators. Two of these operators were directly considered and some definite polarisation states used to get the corresponding positivity bounds, and subsequently compared to publications on experimental data. It shall be noted that in the literature several different conventions and notations are used. For instance, depending on whether authors use the $(+---)$ or $(-+++)$ metric tensor signature, some signs of the coupling constants in the bound will differ. However, it is usually not all terms that are affected, but only those terms in the Lagrangian that have an odd number of index contractions. Other conventions include $F_\mu^\nu F_\nu^\mu$, such as in [15], which are slightly ambiguous and may or may be taken to be[1]. Hence when comparing the signs of the bounds in this dissertation, these conventions should be carefully taken into account. Nonetheless, these would only affect the signs in front of the coupling constants. The moduli of the coefficients in the superpositions should be unaffected.

[1]In any case the difference would result in a different sign at most from $F_{\mu\nu}F^{\mu\nu}$

6.1 Outlook

Whilst this project has resulted in useful results in this limited time frame, the calculations and methods might be used for even broader applications and generalisations as well. The two most important assumptions made were the fact that forward scattering ($t = 0$) and elastic scattering could be exploited. Of course in general, these are not necessarily true, and it may be more difficult to compare elastic, forward bounds to experimental results that are based on empiric data from all types of scattering. However, in theory, experimental data could from general scattering at particle colliders could be searched for scattering events that lie in a small angular cone around

Alternatively, one could focus on generalising the bounds, as in [15], to go beyond the forward limit. This requires using the transversitiy basis, opposed to the helicity basis. In this case, it is much more difficult to obtain the theoretical bounds, however, this would require a more simplified experimental data analysis to compare to. Going beyond the forward limit and the transversity basis requires the use of explicit, definite polarisation states, as indefinite states might have too many free parameters to consider.

Moreover, only the lowest-order bounds were considered, though one could calculate any higher derivatives from these bounds as well. Differentiating generally is trivial, so once a specific positivity bound of lower order is known by direct calculation, this can be used to acquire the higher bounds up to any arbitrary order[2]

In addition to scalars (tensors of rank 0) and vectors (tensors of rank 1) it might be possible to investigate EFTs with spin-2 fields that have a much more elaborate, algebraic structure than simple momenta or polarisation contractions. Usually spin-2 fields are described by the Pauli-Fierz action, and are most frequently used in theories of gravity. This shows another application of EFTs and positivity bounds that could be used in both particle physics and gravity. Of course this entire discussion is also closely related to the question on whether it is possible to generically quantise higher spin fields in the most general way and remains an open topic for research [11].

[2]Although as mentioned in chapter 2, the amplitude being of order $O(s^n)$ would bring about *at most* $n - 1$ non-trivial bounds, and an arbitrary number of trivial ones.

Appendix A

Amplitudes for Proca EFT

Starting with the first term L_1 we consider two separate, free propagator and all its permutations. This yields

$$A_1 = -c_1c_2(\frac{s}{2} - m^2 - \xi_{12} - c_3c_4(\frac{s}{2} - m^2) - \xi_{34})$$

$$+(2\frac{c_2c_1 + c_3c_4}{m^2} + 2\frac{c_1c_3 + c_2c_4}{m^2} + 2\frac{c_1c_4 + c_3c_2}{m^2})(\frac{s^2}{4} - m^2 s)$$

The first line is of order $O(s)$ and will therefore vanish in the second derivatives, so only the second line is relevant.

For L_2 we also consider 2 free propagators and see that the result is

$$A_2 = (c_1c_4+c_2c_3)(2m^2-u)+(c_1c_3+c_4c_2)(2m^2-t)+(c_1c_2+c_3c_4)(s-2m^2)+2m^2\Sigma_{i,j}\xi_{i,j}$$

For L_3:

$$A_3^s = \frac{8m^4 g_0}{\Lambda_\phi^4}(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})(-\frac{c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})$$

$$A_3^t = \frac{8m^4 g_0}{\Lambda_\phi^4}(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})(-\frac{c_2c_4}{m^2}(m^2 - \frac{t}{2}) - \xi_{24})$$

$$A_3^u = \frac{8m^4 g_0}{\Lambda_\phi^4}(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})(-\frac{c_2c_3}{m^2}(m^2 - \frac{u}{2}) - \xi_{23})$$

For L_4

$$A_4^s = \frac{8g_1m^2}{\Lambda_\phi^6}c_1c_2(\frac{s^2}{4} - m^2 s)(\frac{-c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})$$

$$A_4^t = \frac{8g_1m^2}{\Lambda_\phi^6}c_2c_4(\frac{s^2}{4} - m^2 s)(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A_4^u = \frac{8g_1m^2}{\Lambda_\phi^6}c_2c_3(\frac{s^2}{4} - m^2s)(\frac{-c_2c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{24})$$

For L_5:

$$A_5^s = \frac{8g_2m^4}{\Lambda_\phi^6}(m^2 - \frac{t}{2})(\frac{-c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A_5^t = \frac{8g_2m^4}{\Lambda_\phi^6}(\frac{s}{2} - m^2)(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})(\frac{-c_2c_4}{m^2}(m^2 - \frac{t}{2}) - \xi_{24})$$

$$A_5^u = \frac{8g_2m^4}{\Lambda_\phi^6}(\frac{s}{2} - m^2)(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})(\frac{-c_2c_3}{m^2}(m^2 - \frac{u}{2}) - \xi_{23})$$

For L_6:

$$A_6^s = \frac{8g_3m^2}{\Lambda_\phi^6}c_3c_4(\frac{s^2}{4} - m^2s)(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A_6^t = \frac{8g_3m^2}{\Lambda_\phi^6}c_2c_4(\frac{s^2}{4} - m^2s)(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A_6^u = \frac{8g_3m^2}{\Lambda_\phi^6}c_2c_3(\frac{s^2}{4} - m^2s)(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})$$

For L_{7a}:

$$A_{7a}^s = \frac{8g_4}{\Lambda_A^4}(m^2 - \frac{u}{2})^2(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})(\frac{-c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})$$

$$A_{7a}^t = \frac{8g_4}{\Lambda_A^4}(m^2 - \frac{u}{2})^2(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})(\frac{-c_2c_4}{m^2}(m^2 - \frac{t}{2}) - \xi_{24})$$

$$A_{7a}^u = \frac{8g_4}{\Lambda_A^4}(m^2 - \frac{t}{2})^2(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})(\frac{-c_2c_3}{m^2}(m^2 - \frac{u}{2}) - \xi_{23})$$

For L_{7b}:

$$A_{7b}^s = \frac{8g_4}{\Lambda_A^4}\frac{c_3c_4}{m^2}(\frac{s^2}{4} - m^2s)(\frac{s}{2} - m^2)(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A_{7b}^t = \frac{8g_4}{\Lambda_A^4}\frac{c_2c_4}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{t}{2})(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A_{7b}^u = \frac{8g_4}{\Lambda_A^4}\frac{c_2c_3}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{u}{2})(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})$$

For L_{7c}:

$$A_{7c}^s = \frac{-8g_4}{\Lambda_A^4}\frac{c_3c_4}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{u}{2})(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A^t_{7c} = \frac{-8g_4}{\Lambda_A^4}\frac{c_2c_4}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{u}{2})(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A^u_{7c} = \frac{-8g_4}{\Lambda_A^4}\frac{c_2c_3}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{t}{2})(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})$$

For L_{7f}:

$$A^s_{7f} = \frac{8g_4}{\Lambda_A^4}(m^2 - \frac{t}{2})^2(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})(\frac{-c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})$$

$$A^t_{7f} = \frac{8g_4}{\Lambda_A^4}(\frac{s}{2} - m^2)^2(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})(\frac{-c_2c_4}{m^2}(m^2 - \frac{t}{2}) - \xi_{24})$$

$$A^u_{7f} = \frac{8g_4}{\Lambda_A^4}(\frac{s}{2} - m^2)^2(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})(\frac{-c_2c_3}{m^2}(m^2 - \frac{u}{2}) - \xi_{23})$$

For L_{7g}:

$$A^s_{7g} = \frac{-8g_4}{\Lambda_A^4}\frac{c_3c_4}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{t}{2})(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A^t_{7g} = \frac{-8g_4}{\Lambda_A^4}\frac{c_2c_4}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{t}{2})(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A^u_{7g} = \frac{-8g_4}{\Lambda_A^4}\frac{c_3c_2}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{t}{2})(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})$$

For L_{7i}:

$$A^s_{7i} = \frac{-8g_4}{\Lambda_A^4}\frac{c_3c_4}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{u}{2})(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A^t_{7i} = \frac{-8g_4}{\Lambda_A^4}\frac{c_2c_4}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{u}{2})(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A^u_{7i} = \frac{-8g_4}{\Lambda_A^4}\frac{c_2c_3}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{t}{2})(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})$$

For L_{7j}:

$$A^s_{7j} = \frac{-8g_4}{\Lambda_A^4}\frac{c_3c_4}{m^2}(\frac{s^2}{4} - m^2s)(m^2 - \frac{t}{2})(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A^t_{7j} = \frac{-8g_4}{\Lambda_A^4}\frac{c_2c_4}{m^2}(\frac{s^2}{4} - m^2s)(\frac{s}{2} - m^2)(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A^u_{7j} = \frac{-8g_4}{\Lambda_A^4}\frac{c_2c_3}{m^2}(\frac{s^2}{4} - m^2s)(\frac{s}{2} - m^2)(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})$$

For L_{7k}:

$$A_{7k} = \frac{24g_4}{\Lambda_A^4}\frac{c_1c_2c_3c_4}{m^4}(\frac{s^2}{4} - m^2 s)^2$$

For L_{7l}:

$$A_{7l}^s = \frac{-8g_4}{\Lambda_A^4}\frac{c_3c_4}{m^2}(\frac{s^2}{4} - m^2 s)(\frac{s}{2} - m^2)(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A_{7l}^t = \frac{-8g_4}{\Lambda_A^4}\frac{c_2c_4}{m^2}(\frac{s^2}{4} - m^2 s)(m^2 - \frac{t}{2})(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A_{7l}^u = \frac{-8g_4}{\Lambda_A^4}\frac{c_2c_3}{m^2}(\frac{s^2}{4} - m^2 s)(m^2 - \frac{u}{2})(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})$$

For L_{8a}:

$$A_{8a}^s = \frac{32g_5}{\Lambda_A^4}(\frac{s}{2} - m^2)^2(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})(\frac{-c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})$$

$$A_{8a}^t = \frac{32g_5}{\Lambda_A^4}(m^2 - \frac{t}{2})^2(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})(\frac{-c_2c_4}{m^2}(m^2 - \frac{t}{2}) - \xi_{24})$$

$$A_{8a}^u = \frac{32g_5}{\Lambda_A^4}(m^2 - \frac{u}{2})^2(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})(\frac{-c_2c_3}{m^2}(m^2 - \frac{u}{2}) - \xi_{23})$$

For L_{8b}:

$$A_{8b} = A_{8b} = A_{8b}^s + A_{8b}^t + A_{8b}^u = \frac{96g_5}{\Lambda_A^4}c_1c_2c_3c_4(\frac{s^2}{4} - m^2 s)^2$$

For L_{8c}:

$$A_{8c}^s = \frac{-32g_5}{\Lambda_A^4}\frac{c_1c_2}{m^2}(\frac{s^2}{4} - m^2 s)(\frac{s}{2} - m^2)(\frac{-c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})$$

$$A_{8c}^t = \frac{-32g_5}{\Lambda_A^4}\frac{c_1c_3}{m^2}(\frac{s^2}{4} - m^2 s)(m^2 - \frac{t}{2})(\frac{-c_2c_4}{m^2}(m^2 - \frac{t}{2}) - \xi_{34})$$

$$A_{8c}^u = \frac{-32g_5}{\Lambda_A^4}\frac{c_1c_4}{m^2}(\frac{s^2}{4} - m^2 s)(m^2 - \frac{u}{2})(\frac{-c_2c_3}{m^2}(m^2 - \frac{u}{2}) - \xi_{23})$$

For L_{8d}:
For L_{9a}:

$$A_{9a}^s = \frac{8g_6m^4}{\Lambda_\phi^6}(m^2 - \frac{t}{2})(\frac{-c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A_{9a}^t = \frac{8g_6m^4}{\Lambda_\phi^6}(\frac{s}{2} - m^2)(\frac{-c_2c_4}{m^2}(m^2 - \frac{t}{2}) - \xi_{24})(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A^u_{9a} = \frac{8g_6m^4}{\Lambda^6_\phi}(m^2 - \frac{u}{2})(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})(\frac{-c_2c_3}{m^2}(m^2 - \frac{u}{2}) - \xi_{23})$$

For L_{9b}:

$$A^s_{9b} = \frac{8g_6m^2}{\Lambda^6_\phi}c_3c_4(\frac{s^2}{4} - m^2s)(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A^t_{9b} = \frac{8g_6m^2}{\Lambda^6_\phi}c_2c_4(\frac{s^2}{4} - m^2s)(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A^u_{9b} = \frac{8g_6m^2}{\Lambda^6_\phi}c_2c_3(\frac{s^2}{4} - m^2s)(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})$$

For L_{10a}:

$$A^s_{10a} = \frac{16g_7m^4}{\Lambda^6_\phi}(\frac{s}{2} - m^2)(\frac{-c_3c_4}{m^2}(\frac{s}{2} - m^2) - \xi_{34})(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A^t_{10a} = \frac{16g_7m^4}{\Lambda^6_\phi}(m^2 - \frac{t}{2})(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})(\frac{-c_2c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{24})$$

$$A^u_{10a} = \frac{16g_7m^4}{\Lambda^6_\phi}(m^2 - \frac{u}{2})(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})(\frac{-c_2c_3}{m^2}(m^2 - \frac{u}{2}) - \xi_{23})$$

For L_{10b}:

$$A^s_{10b} = -\frac{16g_7m^2}{\Lambda^6_\phi}c_3c_4(\frac{s^2}{4} - m^2s)(\frac{-c_1c_2}{m^2}(\frac{s}{2} - m^2) - \xi_{12})$$

$$A^t_{10b} = -\frac{16g_7m^2}{\Lambda^6_\phi}c_2c_4(\frac{s^2}{4} - m^2s)(\frac{-c_1c_3}{m^2}(m^2 - \frac{t}{2}) - \xi_{13})$$

$$A^u_{10b} = -\frac{16g_7m^2}{\Lambda^6_\phi}c_3c_2(\frac{s^2}{4} - m^2s)(\frac{-c_1c_4}{m^2}(m^2 - \frac{u}{2}) - \xi_{14})$$

As the Lagrangian we are interested in is

$$L_{\text{EFT}} = \Sigma_{i\in\{1,2,\ldots,9,10\}}L_i$$

we now sum over all of these amplitudes in every channel and get

$$M = \Sigma_{j=s,t,u}\Sigma_{i\in\{1_j,\ldots,10_j\}}M_i \tag{A.1}$$

Appendix B

Positivity bounds for Proca EFT

Invoking the positivity bound of lowest order yields the following results:

The L_2 term is of order $O(s)$, so

$$\frac{\partial^2 A}{\partial s^2} = 0.$$

Differentiating L_3 twice gives

$$\frac{\partial^2 A}{\partial s^2} = \frac{4g_0}{\Lambda_\phi^4}(\frac{c_1c_2c_3c_4}{2} + 0 + \frac{c_1c_2c_3c_4}{2}) = \frac{8g_0}{\Lambda_\phi^4} c_1c_2c_3c_4$$

For the s, t and u channels of L_4 we obtain

$$\frac{\partial^2 A_s}{\partial s^2} = \frac{-g_1}{\Lambda_\phi^6} c_1c_2 \cdot \left(6c_3c_4s + 4m^2\xi_{34} - 12c_3c_4m^2\right)$$

$$\frac{\partial^2 A_t}{\partial s^2} = \frac{-4g_1m^2}{\Lambda_\phi^6} c_2c_4(c_1c_3 + \xi_{13})$$

$$\frac{\partial^2 A_U}{\partial s^2} = \frac{-4g_1}{\Lambda_\phi^6} c_2c_3(3c_1c_4s + (-8c_1c_4 + 2\xi_{14})m^2))$$

For L_5:

$$\frac{\partial^2 A_5^s}{\partial s^2} = \frac{4c_1c_2c_3c_4g_2m^2}{\Lambda_\phi^6}$$

$$\frac{\partial^2 A_5^t}{\partial s^2} = 0$$

$$\frac{\partial^2 A_5^u}{\partial s^2} = \frac{g_2 \cdot (6c_1c_2c_3c_4s + 4m^2 \cdot (c_1c_4\xi_{23} + c_2c_3\xi_{14} - 7c_1c_2c_3c_4))}{\Lambda_\phi^6}$$

For L_6:

$$\frac{\partial^2 A_5^s}{\partial s^2} = -\frac{c_3c_4g_3 \cdot (6c_1c_2s + 4m^2 \cdot (\xi_{12} - 3c_1c_2))}{\Lambda_\phi^6}$$

$$\frac{\partial^2 A_5^s}{\partial s^2} = -\frac{4c_2c_4g_3m^2 \cdot (\xi_{13} + c_1c_3)}{\Lambda_\phi^6}$$

$$\frac{\partial^2 A_5^s}{\partial s^2} = -\frac{c_2c_3g_3 \cdot (6c_1c_4s + 4m^2 \cdot (\xi_{14} - 5c_1c_4))}{\Lambda_\phi^6}$$

For L_{7a}:

$$\frac{\partial^2 A_{7a}^s}{\partial s^2} = \frac{2g_4 \cdot (3c_1c_2c_3c_4s^2 + 3m^2 \cdot (c_1c_2\xi_{34} + c_3c_4\xi_{12} - 4c_1c_2c_3c_4)\, s)}{\Lambda_A^4 m^4}$$

$$+\frac{2g_4\, (2m^4 \cdot (\xi_{12} - 3c_1c_2)\, \xi_{34} - 6c_3c_4m^4\xi_{12} + 12c_1c_2c_3c_4m^4)}{\Lambda_A^4 m^4}$$

$$\frac{\partial^2 A_{7a}^t}{\partial s^2} = \frac{4g_4 \cdot (\xi_{13} + c_1c_3)\, (\xi_{24} + c_2c_4)\, (s - 2m^2)}{\Lambda_A^4}$$

$$\frac{\partial^2 A_{7a}^u}{\partial s^2} = \frac{4c_1c_2c_3c_4g_4}{\Lambda_A^4}$$

For L_{7b}:

$$\frac{\partial^2 A_5^s}{\partial s^2} = -\frac{c_3c_4g_4 \cdot (6c_1c_2s^2 + 6m^2 \cdot (\xi_{12} - 4c_1c_2)\, s - 4m^4 \cdot (3\xi_{12} - 5c_1c_2))}{\Lambda_A^4 m^4}$$

$$\frac{\partial^2 A_{7b}^t}{\partial s^2} = -\frac{4c_2c_4g_4 \cdot (\xi_{13} + c_1c_3)}{\Lambda_A^4}$$

$$\frac{\partial^2 A_{7b}^u}{\partial s^2} = -\frac{c_2c_3g_4 \cdot (6c_1c_4s^2 + 6m^2 \cdot (\xi_{14} - 4c_1c_4)\, s - 4m^4 \cdot (3\xi_{14} - 5c_1c_4))}{\Lambda_A^4 m^4}$$

For L_{7c}:

$$\frac{\partial^2 A_{7c}^s}{\partial s^2} = \frac{c_3c_4g_4 \cdot (6c_1c_2s^2 + 6m^2 \cdot (\xi_{12} - 4c_1c_2)\, s - 4m^4 \cdot (3\xi_{12} - 5c_1c_2))}{\Lambda_A^4 m^4}$$

$$\frac{\partial^2 A_{7c}^t}{\partial s^2} = \frac{c_2c_4g_4 \cdot (\xi_{13} + c_1c_3)\, (6s - 12m^2)}{\Lambda_A^4 m^2}$$

$$\frac{\partial^2 A_{7c}^u}{\partial s^2} = \frac{c_2c_3g_4 \cdot (6c_1c_4s + 4m^2 \cdot (\xi_{14} - 3c_1c_4))}{\Lambda_A^4 m^2}$$

For L_{7f}:

$$\frac{\partial^2 A_{7f}^s}{\partial s^2} = \frac{4c_1c_2c_3c_4g_4}{\Lambda_A^4}$$

$$\frac{\partial^2 A_{7f}^t}{\partial s^2} = \frac{4g_4 \cdot (\xi_{13} + c_1c_3)(\xi_{24} + c_2c_4)}{\Lambda_A^4}$$

$$\frac{\partial^2 A_{7f}^u}{\partial s^2} = \frac{2g_4 \cdot (3c_1c_2c_3c_4s^2 + 3m^2 \cdot (c_1c_4\xi_{23} + c_2c_3\xi_{14} - 4c_1c_2c_3c_4)\, s)}{\Lambda_A^4 m^4} + \frac{2g_4\,(2m^4 \cdot (\xi_{14} - 3c_1c_4)\,\xi_{23} - 6c_2c_3m^4\xi_{14} + 12c_1c_2c_3c_4m^4)}{\Lambda_A^4 m^4}$$

For L_{7g}:

$$\frac{\partial^2 A_{7g}^s}{\partial s^2} = \frac{c_3c_4g_4 \cdot (6c_1c_2s + 4m^2 \cdot (\xi_{12} - 3c_1c_2))}{\Lambda_A^4 m^2}$$

$$\frac{\partial^2 A_{7g}^t}{\partial s^2} = \frac{4c_2c_4g_4 \cdot (\xi_{13} + c_1c_3)}{\Lambda_A^4}$$

$$\frac{\partial^2 A_{7g}^u}{\partial s^2} = \frac{c_2c_3g_4 \cdot (6c_1c_4s + 4m^2 \cdot (\xi_{14} - 3c_1c_4))}{\Lambda_A^4 m^2}$$

For L_{7i}:

$$\frac{\partial^2 A_{7i}^s}{\partial s^2} = \frac{c_3c_4g_4 \cdot (6c_1c_2s^2 + 6m^2 \cdot (\xi_{12} - 4c_1c_2)\, s - 4m^4 \cdot (3\xi_{12} - 5c_1c_2))}{\Lambda_A^4 m^4}$$

$$\frac{\partial^2 A_{7i}^t}{\partial s^2} = \frac{c_2c_4g_4 \cdot (\xi_{13} + c_1c_3)(6s - 12m^2)}{\Lambda_A^4 m^2}$$

$$\frac{\partial^2 A_{7i}^u}{\partial s^2} = \frac{c_2c_3g_4 \cdot (6c_1c_4s + 4m^2 \cdot (\xi_{14} - 3c_1c_4))}{\Lambda_A^4 m^2}$$

For L_{7j}:

$$\frac{\partial^2 A_{7j}^s}{\partial s^2} = \frac{c_3c_4g_4 \cdot (6c_1c_2s + 4m^2 \cdot (\xi_{12} - 3c_1c_2))}{\Lambda_A^4 m^2}$$

$$\frac{\partial^2 A_{7j}^t}{\partial s^2} = \frac{c_2c_4g_4 \cdot (\xi_{13} + c_1c_3)(6s - 12m^2)}{\Lambda_A^4 m^2}$$

$$\frac{\partial^2 A_{7j}^u}{\partial s^2} = \frac{c_2c_3g_4 \cdot (6c_1c_4s^2 + 6m^2 \cdot (\xi_{14} - 4c_1c_4)\, s - 4m^4 \cdot (3\xi_{14} - 5c_1c_4))}{\Lambda_A^4 m^4}$$

For L_{7k}:

$$\frac{\partial^2 A_{7k}}{\partial s^2} = \frac{6c_1c_2c_3c_4g_4 \cdot (3s^2 - 12m^2s + 8m^4)}{\Lambda_A^4 m^4}$$

For L_{7l}:

$$\frac{\partial^2 A_{7l}^s}{\partial s^2} = \frac{c_3c_4g_4 \cdot (6c_1c_2s^2 + 6m^2 \cdot (\xi_{12} - 4c_1c_2)\, s - 4m^4 \cdot (3\xi_{12} - 5c_1c_2))}{\Lambda_A^4 m^4}$$

$$\frac{\partial^2 A_{7l}^t}{\partial s^2} = \frac{4c_2c_4g_4 \cdot (\xi_{13} + c_1c_3)}{\Lambda_A^4}$$

$$\frac{\partial^2 A^u_{7l}}{\partial s^2} = \frac{c_2 c_3 g_4 \cdot \left(6 c_1 c_4 s^2 + 6 m^2 \cdot (\xi_{14} - 4 c_1 c_4)\, s - 4 m^4 \cdot (3\xi_{14} - 5 c_1 c_4)\right)}{\Lambda_A^4 m^4}$$

For L_{8a}:

$$\frac{\partial^2 A^s_{8a}}{\partial s^2} = \frac{8 g_5 \cdot \left(3 c_1 c_2 c_3 c_4 s^2 + 3 m^2 \cdot (c_1 c_2 \xi_{34} + c_3 c_4 \xi_{12} - 4 c_1 c_2 c_3 c_4)\, s\right)}{\Lambda_A^4 m^4}$$

$$+ \frac{8 g_5 \left(2 m^4 \cdot (\xi_{12} - 3 c_1 c_2)\, \xi_{34} - 6 c_3 c_4 m^4 \xi_{12} + 12 c_1 c_2 c_3 c_4 m^4\right)}{\Lambda_A^4 m^4}$$

$$\frac{\partial^2 A^t_{8a}}{\partial s^2} = 0$$

$$\frac{\partial^2 A^u_{8a}}{\partial s^2} = \frac{8 g_5 \cdot \left(3 c_1 c_2 c_3 c_4 s^2 + 3 m^2 \cdot (c_1 c_4 \xi_{23} + c_2 c_3 \xi_{14} - 4 c_1 c_2 c_3 c_4)\, s\right)}{\Lambda_A^4 m^4}$$

$$+ \frac{8 g_5 \left(2 m^4 \cdot (\xi_{14} - 3 c_1 c_4)\, \xi_{23} - 6 c_2 c_3 m^4 \xi_{14} + 12 c_1 c_2 c_3 c_4 m^4\right)}{\Lambda_A^4 m^4}$$

For L_{8b}:

$$\frac{\partial^2 A^s_{8b}}{\partial s^2} = \frac{24 c_1 c_2 c_3 c_4 g_5 \cdot (3 s^2 - 12 m^2 s + 8 m^4)}{\Lambda_A^4}$$

For L_{8c}:

$$\frac{\partial^2 A^s_{8c}}{\partial s^2} = \frac{4 c_1 c_2 g_5 \cdot \left(6 c_3 c_4 s^2 + 6 m^2 \cdot (\xi_{34} - 4 c_3 c_4)\, s - 4 m^4 \cdot (3\xi_{34} - 5 c_3 c_4)\right)}{\Lambda_A^4 m^4}$$

$$\frac{\partial^2 A^t_{8c}}{\partial s^2} = \frac{16 c_1 c_3 g_5 \cdot (\xi_{34} + c_2 c_4)}{\Lambda_A^4}$$

$$\frac{\partial^2 A^u_{8c}}{\partial s^2} = \frac{4 c_1 c_4 g_5 \cdot \left(6 c_2 c_3 s^2 + 6 m^2 \cdot (\xi_{23} - 4 c_2 c_3)\, s - 4 m^4 \cdot (3\xi_{23} - 5 c_2 c_3)\right)}{\Lambda_A^4 m^4}$$

For L_{9a}:

$$\frac{\partial^2 A^s_{9a}}{\partial s^2} = \frac{4 c_1 c_2 c_3 c_4 g_6 m^2}{\Lambda_\phi^6}$$

$$\frac{\partial^2 A^t_{9a}}{\partial s^2} = 0$$

$$\frac{\partial^2 A^u_{9a}}{\partial s^2} = \frac{g_6 \cdot \left(6 c_1 c_2 c_3 c_4 s + 4 m^2 \cdot (c_1 c_4 \xi_{23} + c_2 c_3 \xi_{14} - 3 c_1 c_2 c_3 c_4)\right)}{\Lambda_\phi^6} l9c$$

For L_{9c}:

$$\frac{\partial^2 A^s_{9c}}{\partial s^2} = -\frac{c_3 c_4 g_6 \cdot \left(6 c_1 c_2 s + 4 m^2 \cdot (\xi_{12} - 3 c_1 c_2)\right)}{\Lambda_\phi^6}$$

$$\frac{\partial^2 A^t_{9c}}{\partial s^2} = -\frac{4 c_2 c_4 g_6 m^2 \cdot (\xi_{13} + c_1 c_3)}{\Lambda_\phi^6}$$

$$\frac{\partial^2 A_{9c}^u}{\partial s^2} = -\frac{c_2 c_3 g_6 \cdot (6c_1 c_4 s + 4m^2 \cdot (\xi_{14} - 3c_1 c_4))}{\Lambda_\phi^6}$$

For L_{10a}:

$$\frac{\partial^2 A_{10a}^s}{\partial s^2} = \frac{2g_7 \cdot (6c_1 c_2 c_3 c_4 s + 4m^2 \cdot (c_1 c_2 \xi_{34} + c_3 c_4 \xi_{12} - 3c_1 c_2 c_3 c_4))}{\Lambda_\phi^6}$$

$$\frac{\partial^2 A_{10a}^t}{\partial s^2} = 0$$

$$\frac{\partial^2 A_{10a}^u}{\partial s^2} = \frac{2g_7 \cdot (6c_1 c_2 c_3 c_4 s + 4m^2 \cdot (c_1 c_4 \xi_{23} + c_2 c_3 \xi_{14} - 3c_1 c_2 c_3 c_4))}{\Lambda_\phi^6}$$

For L_{10b}:

$$\frac{\partial^2 A_{10b}^s}{\partial s^2} = \frac{2c_3 c_4 g_7 \cdot (6c_1 c_2 s + 4m^2 \cdot (\xi_{12} - 3c_1 c_2))}{\Lambda_\phi^6}$$

$$\frac{\partial^2 A_{10b}^t}{\partial s^2} = \frac{8c_2 c_4 g_7 m^2 \cdot (\xi_{13} + c_1 c_3)}{\Lambda_\phi^6}$$

$$\frac{\partial^2 A_{10b}^u}{\partial s^2} = \frac{2c_2 c_3 g_7 \cdot (6c_1 c_4 s + 4m^2 \cdot (\xi_{14} - 3c_1 c_4))}{\Lambda_\phi^6}$$

Now due to linearity

$$\frac{\partial^2 M_{EFT}}{\partial s^2} = \frac{\partial^2 \Sigma_i M_i}{\partial s^2} = \Sigma_i \frac{\partial^2 M_i}{\partial s^2} > 0$$

so all individual terms above have to be added and their sum is positive.

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